24: Derivatives of Circular Functions and Related Rates

Before we begin, remember that we will (almost) always work in radians. **Radians** don't divide the circle into parts; they measure the size of the central angle in a sector of a unit circle with a certain arc length (whew!). Thus, radians really measure the length of an arc.

The central angle measures one radian

![Figure 1 - Radian Measure](image)

In mathematics, the radian is the preferred (and in many cases, only) option. In applied situations, degrees are preferred—although certain applications require that radians be used; which forces conversions at the beginning and end of the problem!

You should be able to convert between degrees and radians. Since radians deal with arc length, then a full circle angle (360°) is equal to a number of radians that is the same as the circumference (2π). The angle part of that is just 2π…just remember that 180° = π (radians), and you'll always be able to set up a proportion to convert between degrees and radians.

**A: Derivatives of Circular Functions**

Let's jump in from first principles! Start with \( y = \sin(x) \)…

\[
y' = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}
\]

(= the sum/difference identities you've learned…)

\[
\lim_{h \to 0} \left[ \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \right]
\]

\[
\lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}
\]

(factor out the "constants") \[ \sin(x) \] \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \left[ \cos(x) \right] \lim_{h \to 0} \frac{\sin(h)}{h} \]. Once again, we've reduced the problem to the point where we have some limits that need investigating. So—look at the graphs!

The graph of \( \frac{\cos(x) - 1}{x} \) around \( x = 0 \):

The graph of \( \frac{\sin(x)}{x} \) around \( x = 0 \):
It appears that \( \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \) and \( \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \); thus,

\[
\sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x) - \frac{d}{dx}[\sin(x)] = \cos(x).
\]

I'll leave it to you to try this with the cosine function! The result will be

\[
\frac{d}{dx}[\cos(x)] = -\sin(x).
\]

For tangent, use the previous two results, the identity \( \tan(x) = \frac{\sin(x)}{\cos(x)} \) and the quotient rule!

\[
\frac{d}{dx}[\tan(x)] = \sec^2(x).
\]

**B: Derivatives of Reciprocal Circular Functions**

You can use either the quotient rule, or the general power rule to find the derivatives of

\[
\sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}, \quad \text{and} \quad \cot(x) = \frac{1}{\tan(x)}.
\]

\[
\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)
\]

\[
\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)
\]

\[
\frac{d}{dx}[\cot(x)] = -\csc^2(x)
\]

**C: Derivatives of Inverse Circular Functions**

**The Inverse Circular Functions**

In order for a function to have an inverse, the function must be one to one. Unfortunately, the circular functions don't have that property! This is a problem…but it can be fixed by restricting the domain of the function. Once we do that, then we can talk about the arc functions—the inverse functions of the circular functions.

If \( f(x) = \sin(x) \) on the interval \( -\frac{\pi}{2}, \frac{\pi}{2} \), then \( f^{-1}(x) = \sin^{-1}(x) = \arcsin(x) \) on the interval \([-1, 1]\). Note that there are still a few people out there who might talk about \( \text{Arcsin}(x) \) along with \( \arcsin(x) \)—the capitalized version isn't a real inverse function; it completely cancels the sine function when composed. This is a handy feature, but a hard one to legitimize.

Here are the other inverse functions:

**Table 1 - Inverse Circular Functions**

<table>
<thead>
<tr>
<th>Inverse Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>arcsine</td>
<td>[-1, 1]</td>
<td>( -\frac{\pi}{2}, \frac{\pi}{2} )</td>
</tr>
<tr>
<td>function</td>
<td>domain</td>
<td>range</td>
</tr>
<tr>
<td>------------------</td>
<td>-------------------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>arccosine</td>
<td>[-1, 1]</td>
<td>[0, π]</td>
</tr>
<tr>
<td>arctangent</td>
<td>( \mathbb{R} )</td>
<td>( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] )</td>
</tr>
</tbody>
</table>

No one argues about these—everyone recognizes them. Sometimes you may encounter inverse functions for the other three functions…you will not ever find them on a calculator, however.

### Derivatives of Inverse Circular Functions

Rewrite the functions, differentiate implicitly, then solve!

\[ y = \arcsin(x) \Rightarrow \sin(y) = x. \] Differentiate: \( \cos(y) \cdot \frac{dy}{dx} = 1 \). This will require an identity to get rid of the \( y \):

\[
\sqrt{1 - \sin^2(y)} \cdot \frac{dy}{dx} = 1 \Rightarrow \sqrt{1 - x^2} \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.
\]

Use a similar process/identity to do cosine and tangent!

In summary:

\[
\frac{d}{dx} \left[ \arcsin(x) \right] = \frac{1}{\sqrt{1 - x^2}}
\]

\[
\frac{d}{dx} \left[ \arccos(x) \right] = -\frac{1}{\sqrt{1 - x^2}}
\]

\[
\frac{d}{dx} \left[ \arctan(x) \right] = \frac{1}{1 + x^2}
\]

### D: Maxima/Minima with Trigonometry

Er…word problems. 'Nuff said.

### E: Related Rates

We know all about relating two variables in equations—what we haven't done is investigate how the rate of change of one variable is related to the rate of change of another variable (or several other variables), with respect to another variable (time).

Did I mention that we're talking about word problems—applications of calculus?

#### The Technique

You're going to need to write an equation that relates the two variables. Then, take the implicit derivative with respect to \( t \) (time). Now you should be able to plug in the given values and solve for the requested item.

Perhaps this should be illustrated with some examples…

#### Examples

[1.] A rock is dropped into a pond, which causes a circular ripple. The ripple moves away from the center (where the rock was dropped) at 1m/s. How fast is the area of the circle changing when the radius is 3m?
Let's inventory what was given in the problem.

The ripple moves away from the center at 1m/s: this is the rate of change of the radius; \( \frac{dr}{dt} \).

How fast is the area of the circle changing…: this is asking for the rate of change in the area;
\[ \frac{dA}{dt} \).

… when the radius is 3m; \( r = 3 \).

So, the variables are radius, area, and time. We need to write an equation that relates the area and the radius. Any ideas?

Of course! \( A = \pi r^2 \). Now that we've got an equation, we need to differentiate with respect to \( t \).
\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} . \]

Now, we're trying to solve for \( \frac{dA}{dt} \): that means we need to plug in value for everything else—
\( r \) and \( \frac{dr}{dt} \).

Joy! They were given as \( \frac{dr}{dt} = 1 \) and \( r = 3 \)!

\[ \frac{dA}{dt} = 2\pi (3)(1) = 6\pi . \]

When the radius is 3m, the area is changing at \( 6\pi \text{m}^2/\text{s} \).

[2.] Gravel is being poured out of a container onto the ground at a rate of \( 2\text{yd}^3/\text{min} \). The gravel forms a cone as it lands, where the height of the cone is about half of the diameter. How fast is the height of the cone rising when the diameter of the pile is 3yd?

\[ \frac{dV}{dt} = 2 , r = 1.5 , h = \frac{1}{2}d = r , \] and we're asked about \( \frac{dh}{dt} \). We need an equation that relates \( V , r \) and \( h \)...

\[ V = \frac{1}{3} \pi r^2 h , \] so \[ \frac{dV}{dt} = \frac{1}{3} \left[ 2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right] . \] Plug in! \[ 2 = \frac{1}{3} \left[ 2(1.5) \frac{dr}{dt} h + (1.5)^2 \frac{dh}{dt} \right] . \]

Oops—what are we going to do about \( \frac{dr}{dt} \) and \( h \)? Well, \( h = r \), so \( h = 1.5 \). What about \( \frac{dr}{dt} \)?

Well, if \( h = r \), then \( \frac{dh}{dt} = \frac{dr}{dt} \) ! So we get \[ 2 = \frac{1}{3} \left[ 2(1.5) \frac{dh}{dt} (1.5) + (1.5)^2 \frac{dh}{dt} \right] . \] Now we just need to solve for \( \frac{dh}{dt} \) …
Factor! \( 2 = \frac{\pi}{\sqrt{2 \left[ (1.5)^2 + (1.5)^2 \right]}} \frac{dh}{dt} \) means \( \frac{6}{\pi \left[ 2(1.5)^2 + (1.5)^2 \right]} = \frac{dh}{dt} \),

\[
\frac{6}{\pi \left[ 2(1.5)^2 + (1.5)^2 \right]} = \frac{dh}{dt}, \quad \frac{6}{3\pi (1.5)^2} = \frac{dh}{dt}, \quad \frac{2}{\pi (1.5)^2} = \frac{dh}{dt}.
\]

That gives an approximate rate of change of 0.2829yd/min.

[3.] Stranded on Tatooine, R2-D2 and C-3PO have decided to split up in the desert. R2-D2 moves due north at 7km/hr, and C-3PO moves due east at 5km/hr. How fast is the distance between them increasing after 2 hours?

Let's let \( n = \) distance north (of R2-D2), \( e = \) distance east (of C-3PO), and \( D = \) distance between the two. So we're given \( \frac{dn}{dt} = 7, \frac{de}{dt} = 5, \) and \( t = 2 \).

The only equation we can write that relates these is \( n^2 + e^2 = D^2 \). Of course, that means

\[
2n \frac{dn}{dt} + 2e \frac{de}{dt} = 2D \frac{dD}{dt}.
\]

What are we going to plug in for \( n, e \) and \( D \)? How will we use the value \( t = 2 \)?

Of course! At \( t = 2, n = 14, e = 10, \) and \( D = \sqrt{14^2 + 10^2} = \sqrt{196 + 100} = \sqrt{296} \).

\[
2(14)(7) + 2(10)(5) = 2\sqrt{296} \frac{dD}{dt}, \quad \text{so} \quad \frac{dD}{dt} = \frac{296}{2\sqrt{296}} = \frac{148}{\sqrt{296}}, \quad \text{or approximately} \ 8.602\text{km/hr}.
\]