Chapter 15: Probability Rules

OK—time to hunker down and put some detail on the work we started in the previous chapter.

Unions

The **Union** of two events contains all outcomes that belong to either event (and maybe both).

One nice way to think about the union is to use a Venn Diagram. In such a diagram, the universe is depicted as a rectangle, outcomes are points within the rectangle, and events are shapes inside of the rectangle (commonly circles). Here's a Venn Diagram of the union (in red):

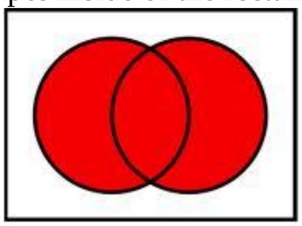

Figure 1 - The Union of Two Events

The word "or" gets used a lot when talking about unions, but it's not being used in the same way that most people use it in conversational English. Outside of statistics class, when you use the word "or", you almost always are excluding the possibility of both things *("do you want to go to the movies or do you want to go shopping?" usually means that only one of the two is an option*). The conversational "or" is (usually) an *exclusive or* (abbreviated *xor*), but the one we use in statistics is an *inclusive or*.

To find the probability of the union, you need to sum up the outcomes in the two events. Note, though, that if you just add the number of outcomes in the first event to the number of outcomes in the second event, then you might be adding some outcomes twice—in the diagram above, that means adding the outcomes (area) of that almond-shaped middle part twice. Thus, to properly add up the outcomes in the union, you need to adjust by subtracting the number of outcomes that belong to *both* events.

Equation 1 - The Probability of the Union

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notice the $\cup$ symbol for the Union. The $\cap$ symbol represents that shared middle area…which we've already talked about in general, and we'll deal with specifically after a few examples.

By the way…don't get to hung up on the symbols and notation. You can probably do most of this without the notation, and the AP Statistics Exam isn't terribly interested in testing your knowledge of probability notation.

Examples

[1.] A bag holds five marbles—one each of red, orange, yellow, green and blue. You reach in a draw out two marbles. What's the probability that you get either a red or a green marble?

This is one where you should probably just list the universe, rather than trying to use a formula. The universe is {ro, ry, rg, rb, oy, og, ob, yg, yb, gb}. In that universe, there are seven outcomes with either a red or a green. Since there are ten outcomes in the universe, the probability is $\frac{7}{10} = 0.7$.

[2.] According to the 2010 US Census, 6.7% of the population is aged 10 to 14 years, and 7.1% of the population is aged 15 to 19 years. If you pick a person at random, what's the probability that they are aged 10 to 19 years?

Just add! Happily, there is no possibility of being in both groups at once, so nothing needs to be subtracted back off. The probability is 13.8%.

[3.] Roll two dice—one red and one green. What's the probability that you roll either a red one or a green five?

There are six ways that the red die can land as a one, and there are six ways that the green die can land as a five—but there is one way that gets counted twice (a red one and a green five). Thus, there are $6 + 6 - 1 = 11$ ways for this to happen, in a universe of size 36…so the probability is $\frac{11}{36}$.

[4.] The hair and eye color of a group of women were measured. The results are shown below.

Table 1 - Hair and Eye Color Data

Hair↓ Eye→	**Brown**	**Blue**	**Hazel**	**Green**
Black	36	9	5	2
Brown	66	34	29	14
Red	16	7	7	7
Blonde	4	64	5	8

One of these women is randomly selected. What's the probability that she either has red hair or green eyes?

There are 37 women with red hair, and 31 women with green eyes—but that accidentally counts 7 of the women twice! There are $37 + 31 - 7 = 61$ women with either red hair or green eyes, in a universe of size 313, so the probability is $\frac{61}{313}$.

Intersections

The **Intersection** of two events contains all outcomes that are shared by the events (that are in common). Here's the Venn Diagram of that:

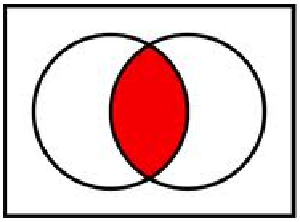

Figure 2 - The Intersection of Two Events

The probability of the intersection is called the **Joint Probability**—two events that do not have any outcomes in common are called **disjoint**, or **mutually exclusive**.

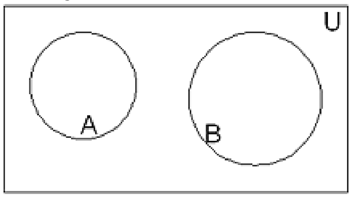

Figure 3 - Two Disjoint Events

Finding the number of outcomes that are shared isn't easy…and thus finding the probability of the intersection isn't as easy as finding the probability of the union. In the last chapter, I mentioned that you could multiply if you knew that the events were unrelated. Let's expand on that and use the correct words now: you can multiply to find the probability of the intersection **only** if you know that the events are **independent**.

Independence means that the knowledge that one event has occurred does not change the probability that the other event will occur. Sometimes, it's obvious that two things must be independent—other times, we'll have to check. More on independence later (after a few examples).

Equation 2 - The Probability of the Intersection of Independent Events

$$P(A \cap B) = P(A) \cdot P(B)$$

Examples

[5.] A bag holds five marbles—one each of red, orange, yellow, green and blue. You reach in a draw out two marbles. What's the probability that you get a red and a green marble?

Again, don't use a formula for this one—just count. There is one outcome out of the ten where you get both a red and a green…so the probability is $\frac{1}{10} = 0.1$.

[6.] Go back to the hair and eye color example (#4). If one of the women is randomly selected, what is the probability that she has red hair and green eyes?

There are 7 women with red hair and green eyes, out of the 313 total women…so the probability is $\frac{7}{313}$.

[7.] Roll two dice—one red and one green. What's the probability that you roll a red one and a green five?

There's a $\frac{1}{6}$ chance that the red die is a one, and the same chance that the green die is a five. The two dice should not influence one another, so the probability of a red one and a green five is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

[8.] A coin is flipped and a die is rolled. What is the probability that the result is heads and a six?

Since the die roll should not affect the coin flip, we can multiply. There is a $\frac{1}{2}$ chance of heads, and a $\frac{1}{6}$ chance of a six, so the probability that they both happen is $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$.

[9.] Bob is at a picnic, but he's in a hurry, so he's not paying much attention as he selects items for his meal. The cooks have made hotdogs and bratwurst (about 70% hotdogs and the rest bratwurst)—Bob just grabs something from the pan as he passes. There is a box full of single serving bags of chips—regular, baked and barbeque in equal quantities. Bob grabs one at random. The drink cooler is evenly split between regular and diet sodas.

What is the probability that Bob selects bratwurst, barbeque chips and a regular soda?

Assuming that each choice is independent of the others, we can just multiply…that results in $\left(\frac{3}{10}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{20} = 0.05$.

Conditional Probability

The **Conditional Probability of an Event** is the probability that one event *will* happen, if it is known that another event *has* happened.

Knowledge is power! What you know affects probability. Two different people, knowing two different things will come up with two different probabilities…and they are both right.

(*the reason why you're having trouble with that concept is your belief in an objective reality…Einstein pretty much ruled out that possibility back in the early 20th century. All measurements are dependent on a frame of reference—an observer*)

When you know that something has happened, that changes the universe—specifically, it shrinks to the given event. How much of the other (non-given) event is left? The intersection, of course! Thus, the size of the other (non-given) event becomes the intersection, and the size of the universe becomes the given event.

Equation 3 - Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Once you have that formula, you can now write a new formula—a more general formula—for the intersection of two events:

Equation 4 - The Probability of Any Intersection

$$P(A \cap B) = P(B) \cdot P(A | B)$$

Examples

[10.] Go back and look at the hair and eye color example again (#4). If we select a woman and find that she is blonde, then what is the probability that she also has blue eyes?

It is given that the woman is blonde—there are 81 such women. Of those, 64 have blue eyes. Thus, the probability that a blonde woman will also have blue eyes is $\frac{64}{81}$.

[11.] As part of an affirmative action lawsuit, a University checks the admission records for students with a particular major. The following was found:

Table 2 - Admissions Data

	Male	Female
Admitted	512	89
Rejected	313	19

(a) What is the probability that a female student will be admitted? In other words, if it is given that the student is female, then what is the probability of admission?

(b) What is the probability that an admitted student is female?

(a) There were 108 female applicants. Of those, 89 were admitted. Thus, the probability that a female applicant will be admitted is $\frac{89}{108}$.

(b) There were 601 student admitted, and 89 of them were female. Thus, the probability that an admitted student will be female is $\frac{81}{601}$.

[12.] 53% of the visitors to a website use Internet Explorer. Of those that use IE, 75% leave a technical support request. Of all of the visitors to this website, what proportion will visit using IE and will leave a technical support request?

The second of those two numbers is actually a conditional probability! You could say $P(IE) = 0.53$ and $P(TS | IE) = 0.75$ are the provided numbers. We've been asked to find $P(IE \cap TS)$…so I can use the general intersection formula! The answer is $(0.53)(0.75) = 0.3975$, or 39.75%.

Independence

Earlier, I wrote that independence means that the knowledge of one event does not affect the probability of another event. In terms of conditional probability, that translates to: events are independent if the **conditional probability of an event is the same as the unconditional** (*regular; usual*) **probability of the event**. Events *A* and *B* are independent if $P(A) = P(A|B)$.

Examples

[13.] One more time—back to the hair and eye color example (#4). Are the events blonde hair and blue eyes independent?

We already know the probability of blue eyes given blonde hair ($\frac{64}{81} \approx 0.7901$), so let's just find the probability of blue eyes. 114 of the 313 women had blue eyes, so the probability is $\frac{114}{313} \approx 0.3642$. Since these probabilities are not equal, these two events are not independent.

[14.] Back to the admissions example (#11)—are the events *female student* and *admitted* independent?

We already know the probability that a female student will be admitted ($\frac{89}{108} \approx 0.8241$), so let's just find the probability of being admitted. 601 of the 933 applicants were admitted, so the probability of being admitted is $\frac{601}{933} \approx 0.6442$. Since these probabilities are not equal, the events are not independent.