

## 7: Applications of Definite Integrals

### 7.1: Integral as Net Change

The **net change** of  $f(x)$  on the interval  $[a, b]$  is  $\int_a^b f(x) dx$ . When talking about motion, this is called **displacement**.

The **total change** in  $f(x)$  on the interval  $[a, b]$  is  $\int_a^b |f(x)| dx$ .

This should sound familiar, as we've already talked about it!

### Examples

[1.] Water flows from a storage tank at the rate of  $R(t) = 200 - 4t$  liters per minute, where  $t \in [0, 50]$ . Find the total amount of water that flows out of the tank in the first ten minutes.

Easy!  $\int_0^{10} (200 - 4t) dt = 200t - 2t^2 \Big|_0^{10} = (200 \cdot 10 - 2 \cdot 10^2) - (0) = 2000 - 200 = 1800$ . A total of 1800 liter of water flow out over the first ten minutes.

[2.] The velocity of a particle moving along the  $x$ -axis is  $v(t) = t^2 - 2t - 8$  meters per second, where  $t \geq 0$ . Find the displacement of the particle over the interval  $[1, 6]$ .

Been there, done that.

$\int_1^6 (t^2 - 2t - 8) dt = \frac{1}{3}t^3 - t^2 - 8t \Big|_1^6 = \left(\frac{1}{3} \cdot 216 - 36 - 8 \cdot 6\right) - \left(\frac{1}{3} - 1 - 8\right) = -\frac{10}{3}$ . The object moved to the left  $\frac{10}{3}$  meters.

### 7.2: Areas in the Plane

#### Areas Between Curves

We know that if  $f(x)$  is above the  $x$ -axis on the interval  $[a, b]$ , then  $\int_a^b f(x) dx$  gives the area between the function and the axis on that interval. We have, on occasion, found more complicated areas by subtracting integrals. Since  $\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b [f(x) - g(x)] dx$ , we can (more efficiently) find the area between two functions by taking the integral of the difference function.

The area between the functions  $f(x)$  and  $g(x)$  on the interval  $[a, b]$  is  $\int_a^b [f(x) - g(x)] dx$ .

## Changing Boundaries

Sometimes, regions don't have constant upper and lower boundaries. Thus, it may be necessary to split the area into regions with constant boundaries—thus, using multiple integrals to find the area of the region.

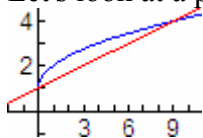
## Integrating with Respect to $y$

We have focused on up-and-down rectangles, since we are most familiar with functions that take an  $x$ -value and produce a  $y$ -value. However, there really isn't any restriction on this! There are times when switching the variable—making the function left-and-right; taking a  $y$ -value to produce an  $x$ -value—makes things easier.

## Examples

[3.] Find the area enclosed by the graphs of  $y = 1 + \sqrt{x}$  and  $y = \frac{3+x}{3}$ .

Let's look at a picture first.



A little algebra will show that these functions intersect at  $x = 0$  and  $x = 9$ .

The area between them is

$$\int_0^9 \left[ 1 + \sqrt{x} - \frac{3+x}{3} \right] dx = x - \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{6}x^2 \Big|_0^9 = \left( 9 - \frac{2}{3}9^{\frac{3}{2}} - \frac{1}{6}9^2 \right) - (0) = \frac{9}{2}.$$

[4.] Find the area enclosed by  $y = x^2$ ,  $y = 2 \cos(x)$ , and the  $x$ -axis.

A picture is worth a thousand words. I've shaded the area of interest.

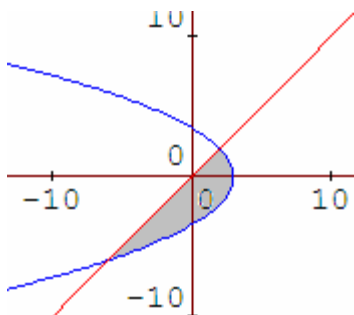


This does not have a constant top—we'll have to find where they intersect, and split the problem into two integrals. I think this is a calculator problem!

$$\int_0^{1.022} x^2 dx + \int_{1.022}^{\frac{\pi}{2}} 2 \cos(x) dx = 0.650.$$

[5.] Find the area in the first quadrant enclosed by the graphs of  $4x + y^2 = 12$  and  $y = x$ .

Making a picture for this may be a bit more involved...again, I've shaded the desired region.



Solving the first equation for  $y$  may be difficult...so don't! Solve it for  $x$ , and we'll integrate with respect to  $y$ . We'll also need to know the  $y$ -coordinates of the intersections...

$$4x + y^2 = 12 \Rightarrow 4x = 12 - y^2 \Rightarrow x = 3 - \frac{y^2}{4}; \quad y = x \Rightarrow x = y. \text{ The points of intersection are}$$

$(2, 2)$  and  $(-6, -6)$ .

The area of the region is

$$\int_{-6}^2 \left[ 3 - \frac{y^2}{4} - y \right] dy = 3y - \frac{1}{12}y^3 - \frac{1}{2}y^2 \Big|_{-6}^2 = \left( 3 \cdot 2 - \frac{1}{12}2^3 - \frac{1}{2}2^2 \right) - \left( 3(-6) - \frac{1}{12}(-6)^3 - \frac{1}{2}(-6)^2 \right) = \frac{64}{3}$$

## 7.3: Volumes

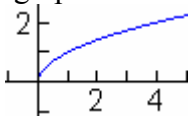
### The Idea

There are many shapes that are created by rotating a simple shape in space. It turns out that the volume of such objects can be found using calculus.

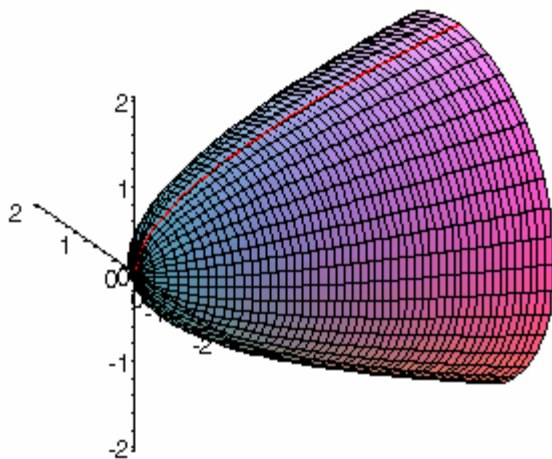
Here's the idea: take a small interval along the axis of rotation, and use that to create a rectangle (much like we did when beginning integrals). Now, spin that rectangle around the axis to create a very small cylinder. Knowing the volume of that cylinder, and being able to add up a lot of those cylinders means that we can find the volume of a solid that is generated by rotating a function around an axis.

### The Disk Method

Let's consider a function (it doesn't matter which—call it  $f(x)$ ) from  $x = 0$  to  $x = 4$ . Here's the graph:

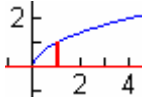


When this is rotated around the  $x$ -axis, the resulting shape is...

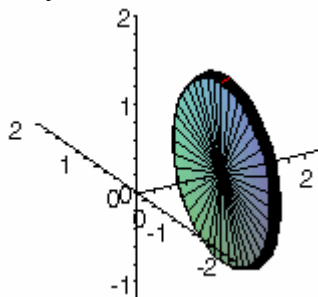


How do we get the volume of this thing? By slicing it up into manageable parts, and taking the limit.

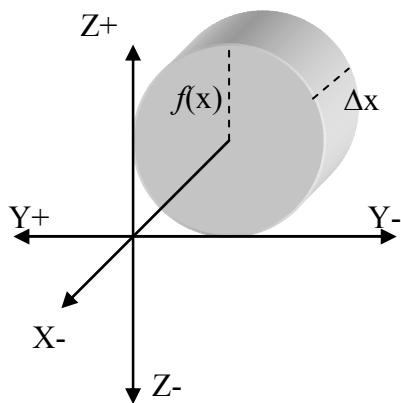
OK, let's consider a really tiny sliver along the  $x$ -axis and make a little rectangle...like this:



If we take this little rectangle, and spin it around the  $x$ -axis, we get a little cylinder (really short cylinders are called disks):



The volume of a cylinder/disk is simple to calculate:  $V = \pi r^2 h$ . The radius is just the function value ( $f(x)$ ), and the height is tiny (let's call it  $\Delta x$ ).



So—the volume of this small cylinder is  $\pi(f(x))^2 \Delta x$ . Now, I want to add up the volumes of a bunch of these disks (specifically, from  $x = 0$  to  $x = 4$ ) and take the limit as  $\Delta x$  approaches zero. That should give the volume of this object!  $V = \lim_{\Delta x \rightarrow 0} \sum_{x=0}^4 \pi(f(x))^2 \cdot \Delta x$

Now wait just a minute...this looks familiar. Haven't we done this before?  
 Yes! That looks like an integral!  
 And it is.

The volume of the object is  $V = \int_0^4 \pi(f(x))^2 dx$ . Of course, you don't always have to start at zero and stop at four...you can use any reasonable limits for the integration.

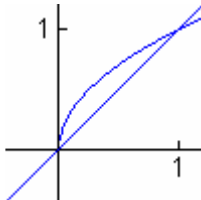
The volume generated by rotating the region of  $f(x)$  on the interval  $[a, b]$  about the  $x$ -axis is  $\pi \int_a^b [f(x)]^2 dx$ .

This example assumes a rotation around the  $x$ -axis. If you want to rotate around the  $y$ -axis, then you probably want to rewrite the function so that  $x$  is a function of  $y$ .

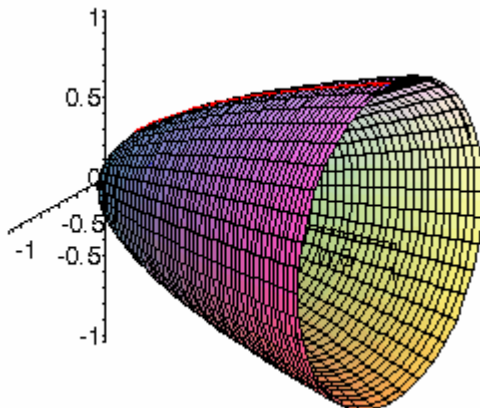
## The Washer Method

Sometimes, the area that you want to rotate doesn't come out solid—it has a hole of some type. This arises when the area that is rotated is the area between two functions, rather than the area beneath just one function.

Consider these functions:

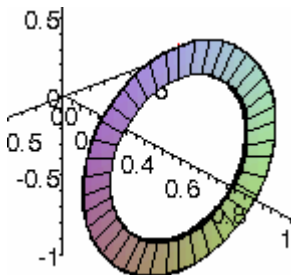


When that very unusual area is rotated, it looks like this:

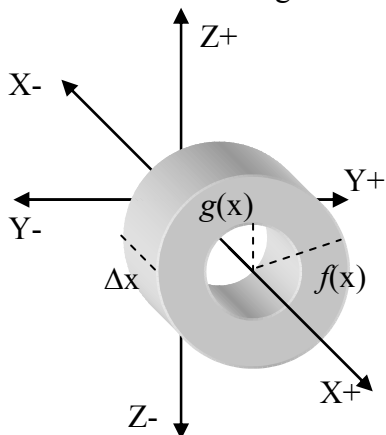


(it is difficult to see, perhaps, that this thing is empty inside...try!)

We can still consider a very small section along the axis to create a rectangle. However, when this is rotated around the axis, you get what most would recognize as a ring (often called a washer)!



The volume of this ring is not difficult to find...



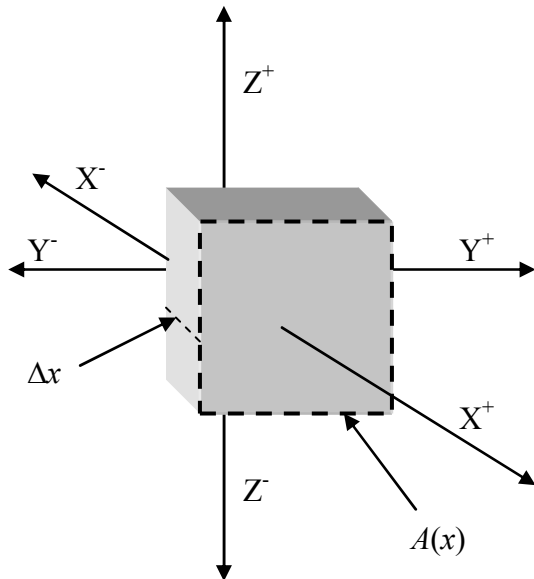
It's just the volume of the disk with the larger radius, minus the volume of the disk with the smaller radius.  $V = \pi(f(x))^2 \Delta x - \pi(g(x))^2 \Delta x = \pi[(f(x))^2 - (g(x))^2] \Delta x$

Again, we want to sum these up and take the limit as  $\Delta x$  approaches zero...and again, this results in an integral!

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2] dx$$

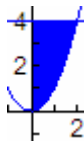
## Cross Sections

In general, if we know the cross sectional area of a solid at any point along an axis (typically the  $x$ -axis), then we can find the volume of the object by integrating the area function along the axis.

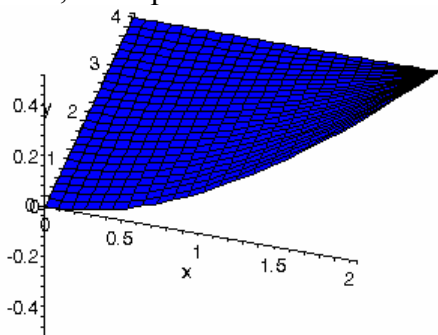


If, for any value of  $x$  in the interval  $[a, b]$  the cross sectional area  $A(x)$  is known (or can be found), then the volume of the object is  $\int_a^b A(x) dx$ .

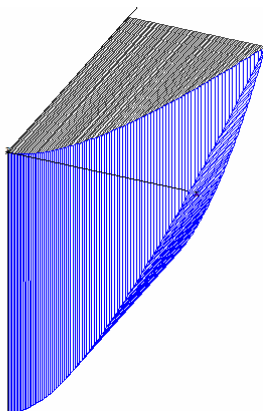
For many problems, we will know a function that traces the base of the solid, and it will be up to us to use a little geometry to find the cross sectional area. For example, consider the area between the  $y$ -axis,  $y = 4$  and  $y = x^2$ .



Now, let's tip that over into three dimensions before we create a solid.



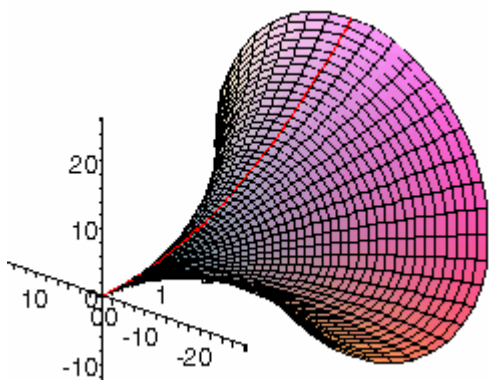
To create the solid, let's construct a square for each value of  $x$ .



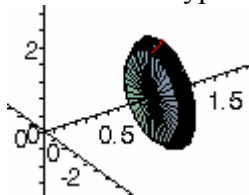
(The squares are shown heading down into the negative  $z$  region.)  
 All that remains is to write the integral...I'll leave that for the examples.

## Examples

[6.] Find the volume of the solid formed when  $y = x^2$  ( $x \in [0, 5]$ ) is rotated about the  $x$ -axis.  
 Here's what that looks like:



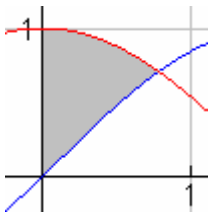
And here's a typical disk formed by that rotation:



The volume is  $\int_0^5 \pi (x^2)^2 dx = \int_0^5 \pi x^4 dx = \frac{\pi}{5} x^5 \Big|_0^5 = \frac{\pi}{5} (3125 - 0) = 625\pi$ .

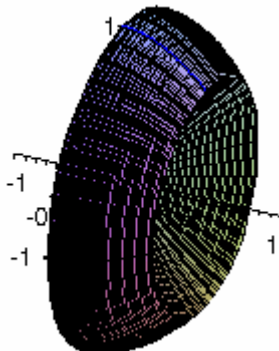
[7.] Find the volume of the solid formed by rotating the region bounded by  $y = \sin(x)$  and  $y = \cos(x)$  (where  $x > 0$ ) around the  $x$ -axis.

First of all, what does that region look like?

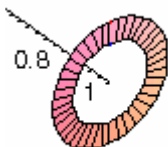


Note that  $y = \cos(x)$  is the function in red, and that the limits on  $x$  are  $x \in \left[0, \frac{\pi}{4}\right]$ .

Now, the volume of revolution:



(looks like a dog's water bowl to me). Finally, a typical ring:



So the volume is  $\int_0^{\frac{\pi}{4}} \pi (\cos^2 x - \sin^2 x) dx$ . But, in order to actually work that out, you're going

to have to use an identity!  $\cos(2x) = \cos^2 x - \sin^2 x$  in particular. So the volume is

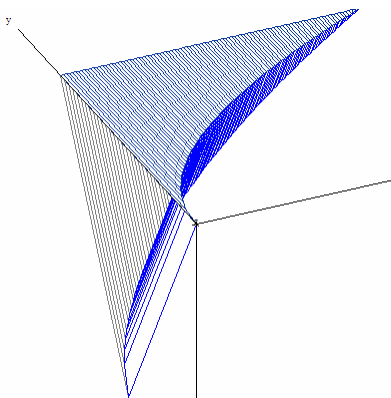
$$\int_0^{\frac{\pi}{4}} \pi \cos(2x) dx = \frac{\pi}{2} \sin(2x) \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2} \left( \sin \frac{\pi}{2} - \sin 0 \right) = \frac{\pi}{2}.$$

[8.] A solid has as its base the region enclosed by  $y = \sqrt{x}$  and  $y = 1$ . Cross sections perpendicular to the  $x$ -axis are equilateral triangles. Find the volume of the solid.

First, the base:



Now, an image of the solid:



(This image is from the left side, so that you can more easily see the equilateral triangles at each  $x$ -value. The axes are the positive  $x$ , positive  $y$ , and negative  $z$ )

The area of an equilateral triangle is  $\frac{\sqrt{3}}{4}b^2$ , where  $b$  is the length of the base.

The volume is

$$\int_0^1 \frac{\sqrt{3}}{4} (1-\sqrt{x})^2 dx = \frac{\sqrt{3}}{4} \int_0^1 (1-2\sqrt{x}+x) dx = \frac{\sqrt{3}}{4} \left[ x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^1 = \frac{\sqrt{3}}{4} \left[ 1 - \frac{4}{3} + \frac{1}{2} \right] = \frac{\sqrt{3}}{24}.$$

## 7.5: Applications from Science and Statistics

Word problems. Fun!

Of course, we all know that work is found by multiplying force applied by distance moved... provided that the force is constant. If it isn't, then we need to find the amount of work done in very small intervals (where the force is almost constant), and add those work bits up.

What do we get when we add up a bunch of little multiplied bits?

(come on... you can get it!)

The same idea applies to fluid pressure—the pressure exerted by a fluid at a depth of  $h$  is  $p = wh$ , where  $w$  is the weight-density of the fluid. If we want to find the pressure exerted over some interval of depths, then we need to add up a bunch of little pressures (again—how do you do this?).

Note that the force exerted by a fluid is the pressure multiplied by the exposed area...

I'm not going to try to explain the statistics to you—it would take too long just to explain the setup of a problem!

### Examples

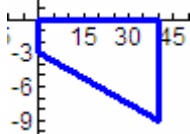
[9.] A particle is propelled along the  $x$ -axis by a force of  $\frac{10}{(1+x)^2}$  Newtons at a distance of  $x$  meters from the origin. How much work is done pushing the particle from  $x=0$  to  $x=9$ ?

The work done is  $\int_0^9 \frac{10}{(1+x)^2} dx$ . Alas, in order to integrate this, a substitution is required. Let  $u = 1+x$  so that  $du = dx$ . When  $x=0$ ,  $u=1$ , and when  $x=9$ ,  $u=10$ . The integral becomes

$\int_1^{10} \frac{10}{u^2} du = 10 \int_1^{10} u^{-2} du = -\frac{10}{u} \Big|_1^{10} = -10 \left( \frac{1}{10} - \frac{1}{1} \right) = -10 \left( -\frac{9}{10} \right) = 9$ . A total of 9 Newton-meters (Joules) of work are done.

[10.] A rectangular swimming pool is 20 feet wide and 40 feet long. The shallow end is 3 feet deep, and the bottom slopes down to a final depth of 9 feet. If the weight-density of water is 62.4 pounds per cubic foot, then how much force is the water exerting on the bottom of the pool?

First, a picture of the longest cross section of the pool.



Let  $x$  be the distance from the shallow end of the pool. The depth at each  $x$  is given by

$h(x) = \frac{3}{20}x + 3$ ; hence the pressure at each  $x$  is  $p(x) = 62.4 \left( \frac{3}{20}x + 3 \right)$ . This pressure is exerted

over a thin ( $\Delta x$ ) 20 foot strip of pool bottom, making the force on that strip

$62.4 \left( \frac{3}{20}x + 3 \right) (20) \Delta x$ . The total force across all such strips is  $\int_0^{40} 62.4 \left( \frac{3}{20}x + 3 \right) (20) dx$ . Just for

fun, I'll let the calculator work that out to 299,520 pounds of force.