

6: Differential Equations and Mathematical Modeling

6.1: Antiderivatives and Slope Fields

Differential Equations

A **differential equation** is an equation which includes a derivative. When asked to **solve** a differential equation, you are to find an equation (function) that works when plugged into the differential equation. Since there are an infinite number of equations with the same derivative, there are typically many solutions to a differential equation. Since all of these equations differ by a constant, finding the equation (but not the constant) is called finding the **general solution**. If, however, you have some additional information that will allow you to also find the constant, then you are finding a **specific** (or *particular*) **solution**.

Solving Initial Value Problems

If the additional information is about when the independent variable equals zero, then we call the problem an **initial value problem**.

To solve such a problem, begin by isolating the derivative (this will often already be done). Next, find an antiderivative of the expression on the right—don't forget that unknown constant! Use the given information to solve for the unknown constant, and finally give your answer as an equation or function.

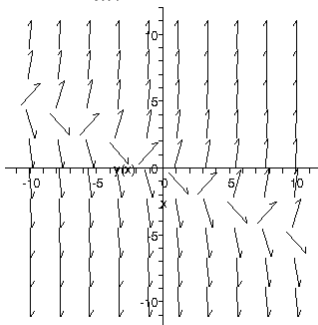
This should be familiar, as we already did some of these a while back...the problem is that you will only be able to do this if the derivative can be expressed solely in terms of the

independent variable— $\frac{dy}{dx} = x^2$ will work; $\frac{dy}{dx} = x + 2y$ will not work!

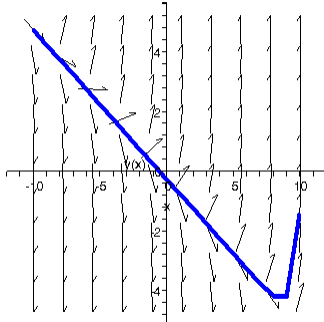
Slope Fields

So—how do you find a solution to a differential equation if you can't find an antiderivative? Use your techno-tools, of course! The given information tells about the slope of some curve at any point in the plane—if we plot those slopes, we might be able to see (or find) an equation that has those slopes at those points.

Such a plot is called a **slope field**. As an example, here is the slope field for the differential equation $\frac{dy}{dx} = x + 2y$:



If we know some additional information about a particular solution—say, the point $\left(-\frac{1}{2}, 5^{-9}\right)$ is on the desired solution—then we can follow the slopes to find the solution.



Pay attention in class when we discuss this!

Antiderivatives and Indefinite Integrals

The process of finding the general solution to a differential equation is also called finding the **indefinite integral**. We denote this with an integral, but no numbers (limits of integration) on the integral sign.

If $f'(x) = F(x)$, then $\int F(x) dx = f(x) + C$, where C is some real number.

Properties of Indefinite Integrals

Constants can be factored out; sums and differences can be split.

Any property of definite integrals that doesn't involve the limits of integration also holds for indefinite integrals.

Examples

[1.] Evaluate $\int (x^3 + 6x + 1) dx$.

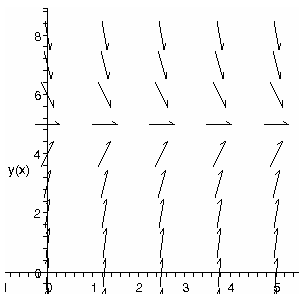
$$\int (x^3 + 6x + 1) dx = \frac{1}{4}x^4 + 3x^2 + x + C$$

[2.] Solve $\frac{dy}{dt} = t^2$ if $y(0) = 2$.

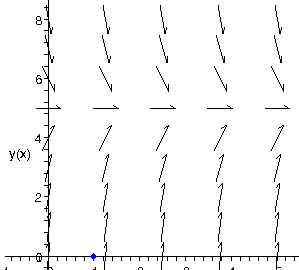
$$\frac{dy}{dt} = t^2 \Rightarrow y = \int t^2 dt = \frac{1}{3}t^3 + C; \text{ since } y = 2 \text{ when } t = 0, 2 = \frac{1}{3}0^3 + C \Rightarrow C = 2. \text{ Thus, the}$$

solution is $y = \frac{1}{3}t^3 + 2$.

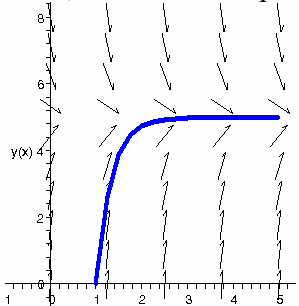
[3.] The slope field for the differential equation $\frac{dy}{dx} = 15 - 3y$ is shown below. Sketch a possible solution curve if it is known that $y(1) = 0$.



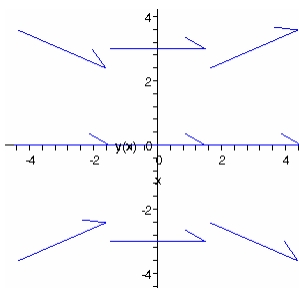
First of all, plot the given point:



Now, follow the slopes:



[4.] Plot nine slope segments (at the integers; centered on the origin) for the slope field of $\frac{dy}{dx} = x \cdot \sin(y)$.



6.2: Integration by Substitution

The Power Rule for Integrals

One of the first things that we did for derivatives was to prove that $\frac{d}{dx}[x^n] = nx^{n-1}$... why not do the same thing for integrals?

If taking the derivative reduces the exponent by one, then integrating should *increase* it by one. Thus, $\int x^n dx$ ought to include x^{n+1} . However, $\frac{d}{dx}[x^{n+1}] = (n+1)x^n$; there's a coefficient there that we don't want. How will we get rid of it?

Divide, of course!

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

This works for more than just x —it works for a power of any function whose derivative (like x) is 1.

Trigonometric Integrands

A quick note—if the integrand is a trigonometric function, then you just have to memorize its integral (just like you had to memorize the derivatives).

Substitution in Indefinite Integrals

Okay—back to the general power rule. I said that it works for powers of functions whose derivative is 1... what about other functions, whose derivatives are not 1?

Recall that $\frac{d}{dx}[f(x)^n] = n \cdot f(x)^{n-1} \cdot f'(x)$. Hence, $\int n \cdot f(x)^{n-1} \cdot f'(x) dx = f(x)^n + C$.

That's a little ugly—let's focus in on the important part. When working forward, you have to use the chain rule to finish up the derivative. When working backwards (doing integrals), the chain rule part has to already be there!

Unfortunately, this can be difficult to see...

Fortunately, there is a method! It's called u -substitution (by most people). The idea is to change variables so that the chain rule part gets "sucked up" into a function whose derivative is 1.

Let's say it this way: To find $\int f(g(x)) \cdot g'(x) dx$, let $u = g(x)$, so that

$\frac{du}{dx} = g'(x) \Rightarrow du = g'(x) dx$. Now swap $g(x)$ for u , and $g'(x) dx$ for du . That makes the integral $\int f(g(x)) \cdot g'(x) dx = \int f(u) du$.

Substitution in Definite Integrals

This substitution—this change of variables—causes problems when the integral is definite. In particular, $\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$.

Separable Differential Equations

A differential equation is **separable** when you can arrange it so that the variables occur exclusively on opposite sides of the equals sign. Separable differential equations can be solved with indefinite integration of both sides of the equation. The other kinds of differential equations have other solution methods... for us, we'll generally use slope fields when we can't separate and integrate.

Examples

[5.] Evaluate $\int 2x(x^2 + 3)^4 dx$.

This is not a simple one—substitution is probably the way to go. So: look for a function raised to a power. I find $x^2 + 3$, so I'll try the substitution $u = x^2 + 3$. The implication of this is that $\frac{du}{dx} = 2x \Rightarrow du = 2x dx$. Substituting: $\int 2x(x^2 + 3)^4 dx = \int u^4 du = \frac{1}{5}u^5 + C$. Now, go back to the original variables: $\frac{1}{5}u^5 + C = \frac{1}{5}(x^2 + 3)^5 + C$.

[6.] Evaluate $\int x^2 \sec(x^3) \tan(x^3) dx$.

This is a little harder to see...let $u = x^3$. That makes $du = 3x^2 dx \Rightarrow \frac{1}{3} du = x^2 dx$. Substituting: $\int x^2 \sec(x^3) \tan(x^3) dx = \int \frac{1}{3} \sec(u) \tan(u) du$. Pull out that coefficient and integrate! You do know what function has this as its derivative, don't you? $\frac{1}{3} \int \sec(u) \tan(u) du = \frac{1}{3} \sec(u) + C$. Finish up with a reverse substitution: $\frac{1}{3} \sec(u) + C = \frac{1}{3} \sec(x^3) + C$.

[7.] Evaluate $\int_0^2 \frac{1}{(2x-3)^2} dx$.

The substitution should be easier to see: $u = 2x - 3 \Rightarrow du = 2 dx \Rightarrow \frac{1}{2} du = dx$. That changes the limits of integration: $x = 0 \Rightarrow u = 2(0) - 3 = -3$, and $x = 2 \Rightarrow u = 2(2) - 3 = 1$. The integral becomes $\frac{1}{2} \int_{-3}^1 \frac{1}{u^2} du$. Work it out: $\frac{1}{2} \int_{-3}^1 u^{-2} du = -\frac{1}{2} u^{-1} \Big|_{-3}^1 = -\frac{1}{2} \left(\frac{1}{1} - \frac{1}{-3} \right) = -\frac{1}{2} - \frac{1}{6} = -\frac{4}{6} = -\frac{2}{3}$.

[8.] Solve $\frac{dy}{dx} = xe^y$ if $y(1) = 0$.

Separate the variables: $\frac{dy}{dx} = xe^y \Rightarrow \frac{dy}{e^y} = x dx \Rightarrow e^{-y} dy = x dx$. Integrate:

$$\int e^{-y} dy = \int x dx \Rightarrow -e^{-y} = \frac{1}{2} x^2 + C. \text{ Apply the initial condition:}$$

$$-e^{-0} = \frac{1}{2}(1)^2 + C \Rightarrow -1 = \frac{1}{2} + C \Rightarrow -\frac{3}{2} = C. \text{ Now solve for } y:$$

$$-e^{-y} = \frac{1}{2} x^2 - \frac{3}{2} \Rightarrow e^{-y} = \frac{3}{2} - \frac{1}{2} x^2 \Rightarrow -y = \ln\left(\frac{3}{2} - \frac{1}{2} x^2\right). \text{ Watch carefully as I simply that a bit:}$$

$$y = -\ln\sqrt{3-x^2}.$$

6.4: Exponential Growth and Decay

Law of Exponential Change

Exponential growth and decay—which you've known and loved for many years—has a connection to calculus. In particular, some quantity y which experiences exponential growth or decay is the solution to the differential equation $\frac{dy}{dt} = ky$.

It is, really! Look: $\frac{dy}{dt} = ky \Rightarrow \frac{1}{y} dy = k dt \Rightarrow \int \frac{1}{y} dy = \int k dt \Rightarrow \ln|y| = kt + C$. Placing the constant of integration on the right side is just a convenience; it doesn't really matter which side it's on. Keep going: $e^{\ln|y|} = e^{kt+C} \Rightarrow |y| = e^{kt} \cdot e^C \Rightarrow y = \pm e^C \cdot e^{kt}$. Now, rather than writing $\pm e^C$, let's just write A . That leaves $y = Ae^{kt}$, where A is the initial value, and k is related to the rate of growth or decay. This is the same old equation that you may well have tattooed on yourself somewhere... anyway, on to some other applications!

Continuously Compounded Interest

This exponential equation works for any continuous process—like continuously compounded interest!

(*Pert*, anyone?)

Radioactivity

...or radioactive decay!

Note that the half-life of a radioactive substance is $\frac{\ln 2}{k}$.

Newton's Law of Cooling

Newton's Law of Cooling says that the rate of change (cooling) of an object is proportional to the difference between the temperature of the object and the ambient (surrounding) temperature:

$\frac{dT}{dt} = -k(T - T_s)$, where T is the temperature of the object and T_s is the ambient temperature.

The solution to this differential equation is $T - T_s = (T_0 - T_s)e^{-kt}$, where T_0 is the initial temperature of the cooling object.

Resistance Proportional to Velocity

If the resistance to a moving object is proportional to the object's velocity, then $m \frac{dv}{dt} = -kv$ (force is proportional to velocity), or $\frac{dv}{dt} = -\frac{kv}{m}$, where m is the mass of the object.

The solution to this differential equation is $v = v_0 e^{-\frac{kt}{m}}$.

Examples

[9.] An *E. Coli* cell, in a nutrient-broth, divides into 2 cells every 20 minutes. If a colony of *E. Coli* begins with 60 cells, then what will the rate of growth be (cells per hour) after 3 hours?

Take a cue from radioactive decay—the doubling time equals $\frac{\ln 2}{k}$; thus

$20 = \frac{\ln 2}{k} \Rightarrow k = \frac{\ln 2}{20}$. Since the colony doubles every 20 minutes, it will have doubled 9 times in 3 hours. That makes the population size $60 \cdot 2^9 = 10240$.

Now we've got what we need! The rate of growth is proportional to the population size; $\frac{dP}{dt} = kP$. Since $k = \frac{\ln 2}{20}$ and $P = 10240$, the rate of growth is $\frac{\ln 2}{20} \cdot 10240 = 512 \cdot \ln(2)$, or approximately 354.891 cells per minute.

[10.] A cup of fresh coffee has a temperature of 95° . When placed in a 20° room, the coffee begins to cool. When the coffee's temperature reaches 70° , it is cooling at the rate of 1° per minute. How long does it take the coffee to reach this point?

To find k , I'll use the information about the rate of cooling.

$\frac{dT}{dt} = -k(T - T_s) \Rightarrow -1 = -k(70 - 20) \Rightarrow k = \frac{1}{50}$. Now I need the solution to the differential equation (since I need to solve for time): $T - T_s = (T_0 - T_s)e^{-kt}$.

$$70 - 20 = (95 - 20)e^{-\frac{1}{50}t} \Rightarrow 50 = 75e^{-\frac{1}{50}t} \Rightarrow \frac{2}{3} = e^{-\frac{1}{50}t} \Rightarrow \ln\left(\frac{2}{3}\right) = -\frac{1}{50}t \Rightarrow t = -50 \ln\left(\frac{2}{3}\right)$$

It takes $-50 \ln\left(\frac{2}{3}\right) \approx 20.273$ minutes.

6.5: Population Growth

Exponential Model

Even though population growth is discrete (you can't gain or lose half a person!), it can typically be modeled with an exponential equation...which is what was done in the previous section.

If the rate of growth is proportional to the population size, then the *relative growth* of the population is constant: $\frac{dP}{dt} = kP \Rightarrow \frac{1}{P} \frac{dP}{dt} = k$.

Logistic Growth Model

The real problem with using an exponential equation to model population growth is that exponential equations *keep growing forever*. This does not ever occur in nature; at some point, the size of the population grows to the point that it uses all of the surrounding resources, and cannot support any additional members. This number is the **carrying capacity**—the largest population size that nature and surrounding resources can support.

Since there is an upper limit, an exponential model doesn't really work in the long run. The rate of growth of a population is proportional to the size of the population relative to the carrying capacity—or, to the relative amount of the carrying capacity remaining: $1 - \frac{P}{M}$. Thus, a better

model is $\frac{1}{P} \frac{dP}{dt} = k \left(1 - \frac{P}{M}\right)$, or $\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)$. This is called a **logistic model**.

This differential equation is separable—with a little (!) work, you can find the general solution to be $\frac{M - P}{P} = Ae^{-kt}$.

Example

[11.] The Pacific halibut fishery is modeled by the differential equation $\frac{dy}{dt} = ky \left(1 - \frac{y}{C}\right)$,

where y is the biomass (total mass of the fish) in kilograms and time is measured in years. The carrying capacity C is estimated to be 80000000, and $k = 0.71$. If the initial biomass is 20000000 kilograms, how long should it take the biomass to reach 40000000 kilograms?

I'll go ahead and use the general solution from above—I just need to find A .

$$\frac{80000000 - 20000000}{20000000} = Ae^{-0.71(0)} \Rightarrow A = 30000000;$$

$$\frac{80000000 - 40000000}{40000000} = 30000000e^{-0.71t} \Rightarrow$$

$$1 = 30000000e^{-0.71t} \Rightarrow \frac{1}{30000000} = e^{-0.71t} \Rightarrow$$

$$\ln\left(\frac{1}{30000000}\right) = -0.71t \Rightarrow t = \frac{-\ln(30000000)}{-0.71} \approx 24.249$$

It should take about 24.249 years for this hatchery to reach a total biomass of 40 million kilograms.

6.6: Numerical Methods

Euler's Method

Sometimes it is difficult to solve a differential equation...many minds have taken to the task of finding ways to solve them. One of mathematics' All-Stars devised a method: Leonhard Euler (Swiss; 1707-1783). His idea was to use the linearization of the solution curve to create a bunch of small line segments that approximate the solution curve.

Given a differential equation $\frac{dy}{dx} = f(x, y)$ and an initial condition $y(x_0) = y_0$, the linearization of the solution curve $y = y(x)$ is

$L(x) = y(x_0) + y'(x - x_0) = y(x_0) + f(x_0, y_0)(x - x_0)$. Note that $f(x_0, y_0)$ is the slope at the point (x_0, y_0) —which is the slope shown in a slope field at that point!

This linearization can be used to approximate a second point on the solution curve (for some value of x which is relatively close to x_0)...which can be used to make another line, which can be used to approximate another point, which can be used to make another line...and so on. Of course, the potential error gets larger the farther away from the initial point you go...

Numerical Solutions

You can automate this line-point-line-point method; this will create a table of points, or a numerical solution to the differential equation.

Example

[12.] Use Euler's method to estimate $y(0.4)$ if $y' = y$ and $y(0) = 1$.

The initial point is $(0,1)$, and the slope of the curve at that point is 1.

The initial line is $y_1 = 1 + 1(x_1 - 0) = 1 + x_1$. I'll choose $x_1 = 0.1$, which makes $y_1 = 1 + 0.1 = 1.1$.

The next point is $(0.1, 1.1)$, and the slope is 1.1. The next line is

$$y_2 = 1.1 + 1.1(x_2 - 0.1) = 0.99 + 1.1x_2. \text{ Let } x_2 = 0.2, \text{ so that}$$

$$y_2 = 0.99 + 1.1(0.2) = 0.99 + 2.2 = 3.19.$$

The next line is $y_3 = 3.19 + 3.19(x_3 - 0.2) = 2.552 + 3.19x_3$. Let $x_3 = 0.3$, which makes

$$y_3 = 3.509. \text{ The final line is } y_4 = 3.509 + 3.509(x_4 - 0.3) = 2.5463 + 3.509x_4. \text{ Allowing } x_4 = 0.4$$

(which was what we were aiming for in the first place!) gives $y_4 = 3.8599$.

Thus, $y(0.4) \approx 3.860$.