5: The Definite Integral

5.1: Estimating with Finite Sums

Consider a moving object…its velocity (meters per second) at any time (seconds) is given by $v(t) = -9.8t + 25$. Can we use this information to determine the distance that the object has traveled during the first four seconds? The old $d = r \cdot t$ only works for constant speeds…

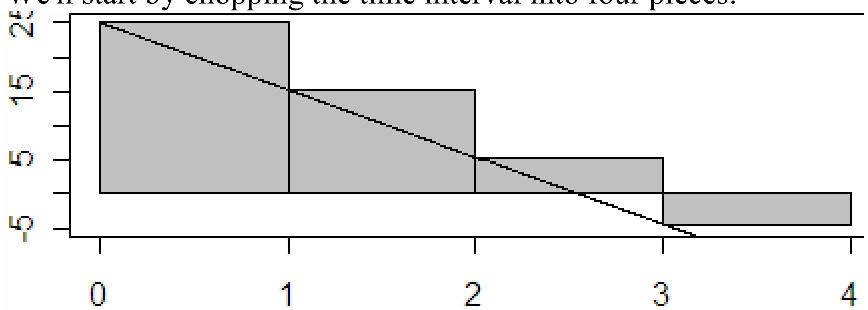
Divide and Conquer

Let's take a cue from the movies! As you know, a movie is just a series of still images, that are run really fast. Each image is still (no movement), but a series of them give the illusion of movement. Perhaps we could take our velocity equation, chop it up into a series of still images, and get the motion from that…

Let's say it this way: for any short interval of time, the change in velocity is not very large—thus, for a short interval of time, we could consider the object to be moving with constant velocity, and use our old familiar distance equation.

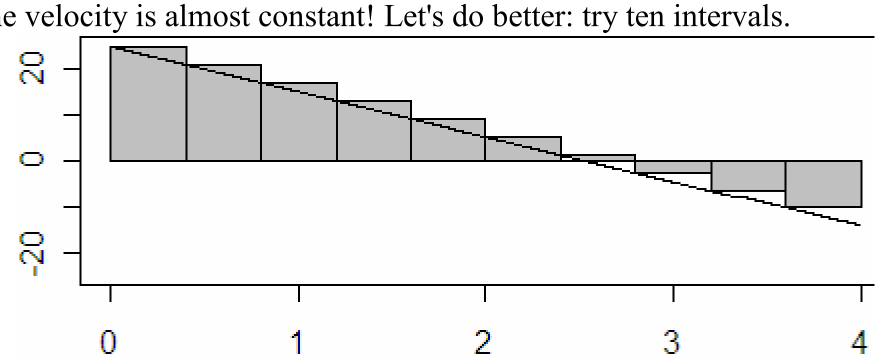
There is a graphic interpretation of this idea—creating rectangles around the graph of $v(t)$.

We'll start by chopping the time interval into four pieces:

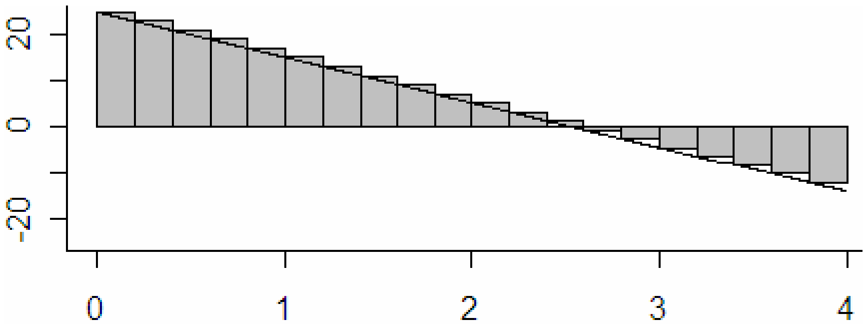


Notice that the object is at rest around time 2.5 seconds—thus, it turns around and heads back in the other direction. Thus, the total distance traveled will be different from the net distance traveled (displacement). Keep this in mind for later!

The total area of these rectangles is 50—the total distance traveled is 50 meters. This is only an approximation, though…we're supposed to be chopping the time interval into sections where the velocity is almost constant! Let's do better: try ten intervals.



That gives a total area of 44.704. This should be a little better…however, there will still be some error! Let's try once again, with 20 intervals:



This gives an area of 43.328…this is better, but still not *the* answer.

Reductio ad infinitum

The answer requires that we consider an infinite number of intervals! However, we'll leave that for a bit, since that requires limits. Let's take one step at a time.

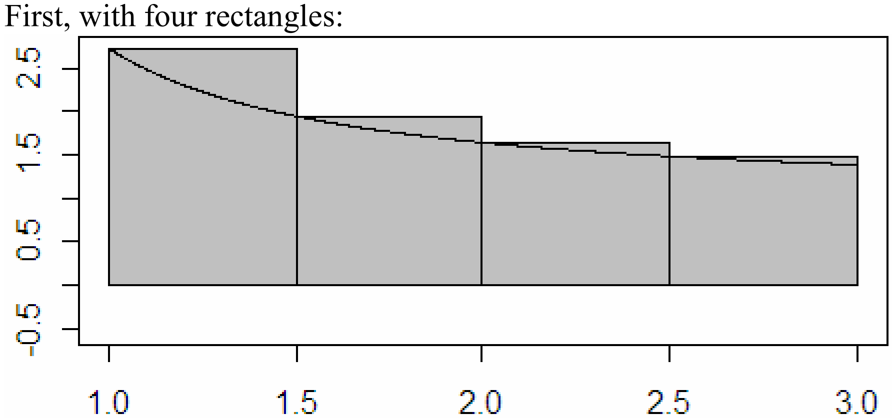
Three Paths to the Summit

The pictures that I drew above all have rectangles that have the left side on the function—we call these left-hand rectangles. You could also use midpoints (the height of the rectangle is at the midpoint of the top edge) or right sides (take a wild guess!). You'll get slightly different answers…but that's okay; once we work towards *the* answer, they'll all come out the same.

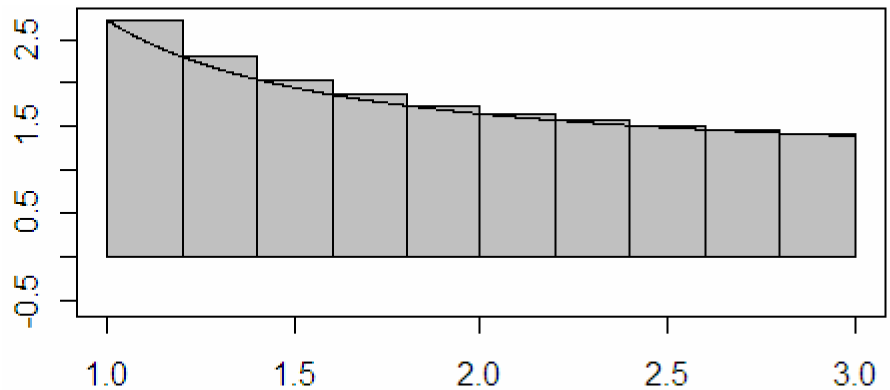
Examples

[1.] Estimate the area under the curve $y = e^{\frac{1}{x}}$ in the interval $[1,3]$ with four and ten rectangles.

First, with four rectangles:



The estimated area is 3.903. Now with ten rectangles:



The estimated area is 3.663.

[2.] The following data were collected from a moving vehicle:

time (sec)	0	5	10	15	20	25	30
velocity (ft/sec)	25	31	35	43	47	46	41

Estimate the distance traveled by the vehicle during the thirty second interval.

Note that the data have already divided the interval into a partition! The width of each subinterval is 5 (seconds), and the height of the rectangle is the given velocity (feet per second).

width	5	5	5	5	5	5	5
height	25	31	35	43	47	46	41
area	125	155	175	215	235	230	205

This gives a total area of 1340—which means a total distance of 1340 feet.

5.2: Definite Integrals

The Generic Path: Riemann Sums

Let's generalize the ideas we explored in the previous section…

A partition of the interval $[a,b]$ is a set of values (x_i) so that $a < x_1 < x_2 < ... < x_{n-1} < b$. If we call $x_0 = a$ and $x_n = b$, then $P = \{x_0, x_1, x_2 ... x_n\}$ is a **partition** of the interval $[a,b]$.

The k^{th} subinterval of *P* is $[x_{k-1}, x_k]$. Let c_k be any value in the k^{th} subinterval of *P*. Also, let Δx_k be the width of the k^{th} subinterval of *P*.

If *f* is a function which is continuous on the interval $[a,b]$, then the **Riemann sum of *f*** (on that interval) is $\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$. Note that this is the sum of a bunch of rectangle areas…

The Definite Integral

Back when we were developing the idea of derivatives, we created a generic point, and then moved the point arbitrarily close to the target point…we allowed the distance between the points to approach zero. This limiting process gave us the slope of the curve.

Now consider the rectangles areas from the Riemann sum. One might think that adding more points to the partition might make the rectangle areas get really small, and add up to the area under the curve…however, just adding more points doesn't make all of the rectangles small. To do that, you have to add points so that the largest Δx_k approaches zero (the largest Δx_k is called the **norm of *P***, or $\|P\|$).

If $\lim\limits_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$ exists, then we say that *f* is **integrable** over the interval $[a,b]$, and the value of the limit is called the **definite integral of *f* over** $[a,b]$.

Notation

That limit notation is terrible difficult to write, so we create a new notation for the definite integral of *f* over $[a,b]$: $\int_a^b f(x)\,dx$. This is read as "the integral of *f* from *a* to *b*."

Evaluating Definite Integrals

Definite integrals are defined in such a way that they approximate the area between a curve and the *x*-axis (if it lies above the *x*-axis). However, since this is math (and not reality), the area between the curve and the *x*-axis for functions which lie below the *x*-axis is *negative*. Thus, the definite integral of *f* over $[a,b]$ is the area above the axis plus the area below the axis.

If you really want area (real, positive area), then it's area above the axis *minus* area below the axis.

Returning to our velocity example from a while back: the total distance traveled will be the area between the line $v(t) = -9.8t + 25$ and the *x*-axis. Fortunately, this is just a pair of triangles! First, note that $v(2.551) = 0$. The area of the left/upper triangle is $\frac{1}{2}(25)(2.551) = 31.888$, and the area of the right/lower triangle is $\frac{1}{2}(-14.2)(4-2.551) = 10.288$, for a total area of 42.176 (a total distance of 42.176 meters). The displacement is $31.888 - 10.288 = 21.600$ meters.

Of course, rather than doing all of this by hand, you could use your calculator…note that the calculator will do the definite integral; not the area under the curve. My calculator gives the value of the definite integral at 21.6.

Examples

[3.] Find $\int_{-1}^{7} 2\,dx$.

This is a rectangle with base length of 8 and a height of 2; the area (and the value of the definite integral) is 16.

[4.] Find $\int_0^5 (5-x)\,dx$.

This is a triangle with base 5 and height 5; the area is $\frac{1}{2}(5)(5) = \frac{25}{2}$.

[5.] Find $\int_{-2}^{2} \sqrt{4-x^2}\, dx$.

This is a semicircle of radius 2. The area is 4π.

5.3: Definite Integrals and Antiderivatives

Properties of the Definite Integral

Reversing the limits negates the integral: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

The integral of a point is zero: $\int_a^a f(x)\,dx = 0$

Constants can be factored out: $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$

Integrals can be "distributed" across sums and differences:

$\int_a^b \left[f(x) \pm g(x)\right] dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Integrals over adjacent intervals can be combined: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

The Average Value of a Function

We all know how to find an average—add up all the values, and divide by the number of values. It turns out that you can do that with an infinite number of things, also! Adding an infinite number of things creates an integral, though.

The average value of $f(x)$ over the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

The Mean Value Theorem (for Definite Integrals)

There is some point ($x = c$) in the interval $[a,b]$ so that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

The Mean Value Theorem for Derivatives says that for any interval, there is some point where the slope of the tangent equals the slope of the secant. The Mean Value Theorem for Integrals says that for any interval, there is some point where the area of a rectangle equals the area under the curve ($(b-a)f(c) = \int_a^b f(x)\,dx$).

The Connection: A First Glimpse

The derivative is found with a quotient of infinitely small differences.

The definite integral is found through a sum of infinitely small products (areas).

Sums and differences…quotients and products…surely there is some connection!

…and there is.

Here's a look ahead—which you'll need for a few of the problems in this section.

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

Examples

[6.] If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, then find $\int_0^9 \left[2f(x)+3g(x)\right]dx$.

Using the properties of integrals,

$$\int_0^9 \left[2f(x)+3g(x)\right]dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37)+3(16) = 122$$

[7.] Find $\int_{-1}^1 \left(x^2 - x - 1\right)dx$.

Since I don't know a handy formula for the area under a parabola, I'll use antiderivatives. One antiderivative of $f(x) = x^2 - x - 1$ is $F(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - x$. Thus, $\int_{-1}^1 f(x)\,dx = F(1) - F(-1)$.

$F(1) = \frac{1}{3} - \frac{1}{2} - 1 = -\frac{7}{6}$ and $F(-1) = -\frac{1}{3} - \frac{1}{2} + 1 = \frac{1}{6}$, so $\int_{-1}^1 \left(x^2 - x - 1\right)dx = -\frac{7}{6} - \frac{1}{6} = -\frac{8}{6} = -\frac{4}{3}$.

[8.] Find the average value of $-\frac{1}{x^2}$ on the interval $[1,4]$.

The formula gives me $\frac{1}{4-1}\int_1^4 -\frac{1}{x^2}\,dx$. I'll have to use an antiderivative again…how about $\frac{1}{x}$?

$$\frac{1}{3}\int_1^4 -\frac{1}{x^2}\,dx = \frac{1}{3}\left[\frac{1}{x}\right]_1^4 = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{3}\left(-\frac{1}{12}\right) = -\frac{1}{36}.$$

5.4: The Fundamental Theorem of Calculus

FTC, Part 1

If $g(x) = \int_a^x f(t)\,dt$, then $\frac{d}{dx}\left[g(x)\right] = f(x)$.

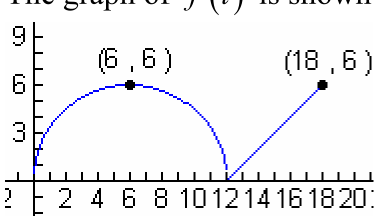
In other words, the derivative of an integral is just the plain function. It's as if the derivative and the integral are inverse operations…

The hairs on the back of your neck should be standing on end…

The Integral as a Function

The idea of $g(x) = \int_a^x f(t)\,dt$ needs some additional investigation…there are lots of potential questions that can be asked about functions that are defined as integrals of other functions! Here are just a few examples from a single scenario…

The graph of $f(t)$ is shown below. It consists of a semicircle and a line segment.



Define $A(x)=\int_0^x f(t)\,dt$.

$A(6)$ is the area under $f(t)$ between $x=0$ and $x=6$—$\int_0^6 f(t)\,dt$—which is a quarter circle. $A(6)=\frac{1}{4}\pi(6)^2=\frac{36\pi}{4}=9\pi$.

$A(18)$ is the area under $f(t)$ between $x=0$ and $x=18$: $\int_0^{18} f(t)\,dt$. This is a semicircle and a triangle! $A(18)=\frac{1}{2}\pi(6)^2+\frac{1}{2}(6)(6)=18\pi+18$.

Note that $A(x)$ is an increasing function—as *x* gets larger, the amount of area under $f(t)$ keeps getting larger.

The average value of $f(t)$ on the interval $[0,18]$ is $\frac{1}{18-0}\int_0^{18} f(t)\,dt=\frac{1}{18}\cdot A(18)=\pi+1$.

FTC, Part 2

If $f(x)$ is any antiderivative of $g(x)$, then $\int_a^b g(x)\,dx=f(b)-f(a)$.

Note that $f(x)$ being an antiderivative of $g(x)$ means that $f'(x)=g(x)$. Thus, this theorem is taking the integral of an antiderivative. Also note that the integral of this derivative resulted in the original function (sort of).

Finding Area

If you want to use definite integrals to find the area between a curve and the *x*-axis, remember that areas below the axis will be negative! One way to find the total area is to first determine where the function is equal to zero—this will divide the areas into above/below the *x*-axis. Add the integrals from the sections above the axis, and subtract the integrals from the sections below the axis.

Another way to do this is to just find the area under the absolute value of the function. The absolute value will simply force all of the areas to be positive. This makes things really easy—especially if you are using your calculator…

Examples

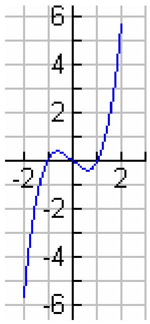
[9.] Find $\int_0^{\pi}\cos(x)\,dx$.

Well…an antiderivative of $\cos(x)$ is $\sin(x)$…so

$\int_0^{\pi}\cos(x)\,dx=\sin(x)]_0^{\pi}=\sin(\pi)-\sin(0)=0-0=0$. Note that this does not mean that the area between $\sin(x)$ and the *x*-axis is zero!

[10.] Find the total area between $f(x)=x^3-x$ and the *x*-axis on the interval $[-2,2]$.

Let's be careful and look at the graph.



We'll need to split this up! The total area will be

$$-\int_{-2}^{-1}\left(x^3-x\right)dx+\int_{-1}^{0}\left(x^3-x\right)dx-\int_{0}^{1}\left(x^3-x\right)dx+\int_{1}^{2}\left(x^3-x\right)dx$$

An antiderivative of $f(x)=x^3-x$ is $F(x)=\frac{1}{4}x^4-\frac{1}{2}x^2$. That gives us

$$-\left[F(x)\right]_{-2}^{-1}+\left[F(x)\right]_{-1}^{0}-\left[F(x)\right]_{0}^{1}+\left[F(x)\right]_{1}^{2}$$

$$-\left[F(-1)-F(-2)\right]+\left[F(0)-F(-1)\right]-\left[F(1)-F(0)\right]+\left[F(2)-F(1)\right]$$

$$-F(-1)+F(-2)+F(0)-F(-1)-F(1)+F(0)+F(2)-F(1)$$

$$F(-2)-2F(-1)+2F(0)-2F(1)+F(2)$$

Let's start plugging in!

$$F(-2)=\frac{1}{4}16-\frac{1}{2}4=4-2=2$$

$$F(-1)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$$

$$F(0)=0$$

$$F(1)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$$

$$F(2)=\frac{1}{4}16-\frac{1}{2}4=4-2=2$$

So finally, we get $2-2\left(-\frac{1}{4}\right)+2(0)-2\left(-\frac{1}{4}\right)+2=2+\frac{1}{2}+\frac{1}{2}+2=5$.

[11.] Find $\frac{d}{dx}\left[\int_{-\pi}^{x}\tan(t)\,dt\right]$.

Easy! The derivative is $\tan(x)$.

5.5: The Trapezoidal Rule

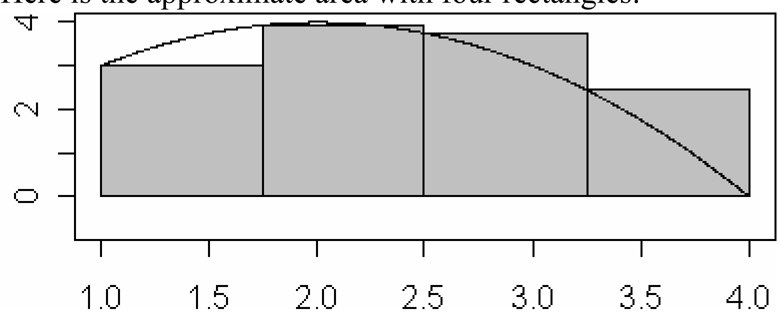
Another Path

Now that we have a firm connection between integrals and area, let's approach the area idea again. Before, we used rectangles to approximate the area under the curve—simply because rectangles are very simple geometric figures. Alas, they don't really do such a good job of approximating area under the curve until you use a *lot* of *very tiny* rectangles.

There is another shape which does a better job—the trapezoid! Divide the interval in question as you would before, but now connect the function values to obtain trapezoids. For our linear example (way back at the beginning of this chapter), we get an exact answer with as few as two trapezoids!

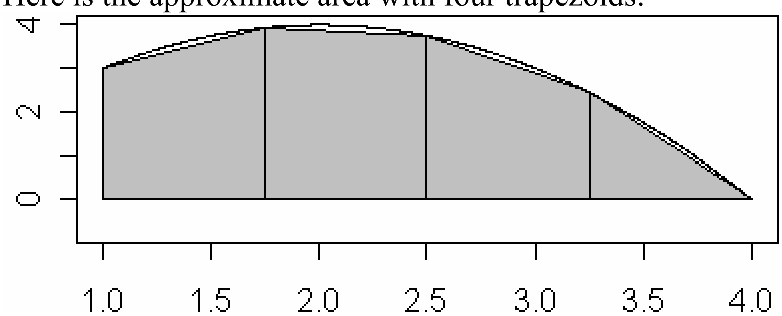
Let's take the function $f(x) = 4 - (x-2)^2$ on the interval $[1,4]$ with four regions as an example.

Here is the approximate area with four rectangles:



(that's an area of 9.8438)

Here is the approximate area with four trapezoids:



(that's an area of 8.7188)

The actual area is 9. You can see that the trapezoids do a much better job of the approximation!

Simpson's Rule/Approximation

You can do even better with your approximation if, instead of trapezoids, you use rectangles capped with little parabolic arches. It turns out that there is a (relatively simple) formula for this—and a (fairly simple) formula that generalizes the entire procedure!

It's called **Simpson's Rule**:

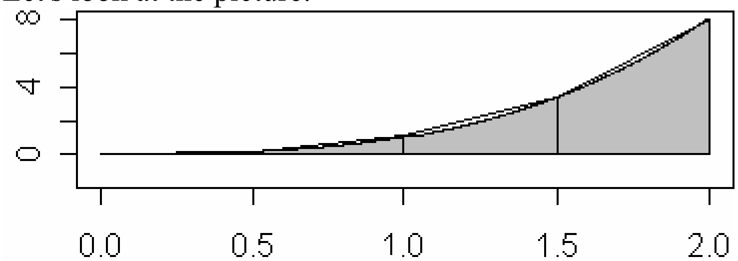
To approximate $\int_a^b f(x)\,dx$, divide the interval $[a,b]$ into an *even* number of subintervals (make n even) and then calculate $\frac{b-a}{3n}\left(y_o + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$.

Happily, this topic is not part of the AP Exam!

Examples

[12.] Use four trapezoids to estimate the value of $\int_0^2 x^3\,dx$.

Let's look at the picture:



The areas of the trapezoids are

$$\frac{1}{2}\left(0^3+\left(\frac{1}{2}\right)^3\right)\frac{1}{2}=\frac{1}{2}\left(\frac{1}{8}\right)\frac{1}{2}=\frac{1}{32}$$

$$\frac{1}{2}\left(\left(\frac{1}{2}\right)^3+1^3\right)\frac{1}{2}=\frac{1}{2}\left(\frac{1}{8}+1\right)\frac{1}{2}=\frac{1}{2}\left(\frac{9}{8}\right)\frac{1}{2}=\frac{9}{32}$$

$$\frac{1}{2}\left(1^3+\left(\frac{3}{2}\right)^3\right)\frac{1}{2}=\frac{1}{2}\left(1+\frac{27}{8}\right)\frac{1}{2}=\frac{1}{2}\left(\frac{35}{8}\right)\frac{1}{2}=\frac{35}{32}$$

$$\frac{1}{2}\left(\left(\frac{3}{2}\right)^3+2^3\right)\frac{1}{2}=\frac{1}{2}\left(\frac{27}{8}+8\right)\frac{1}{2}=\frac{1}{2}\left(\frac{91}{8}\right)\frac{1}{2}=\frac{91}{32}$$

This gives a total area of $\frac{1}{32}+\frac{9}{32}+\frac{35}{32}+\frac{91}{32}=\frac{136}{32}=4.25$.

(*the actual area is 4*)