4: Applications of Derivatives

4.1: Extreme Values of Functions

First of all, *extrema* is the plural of *extremum*—extrema refers to maximum and minimum values.

Global (Absolute) Extrema

A global (or absolute) maximum is the largest value in the range of the function. Technically: $f(c)$ is a global maximum (value) of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain of *f*.

Similarly, a global minimum is the smallest value in the range of a function. I'll let you think about the technical way to say that.

There is no guarantee that a function will actually have a global maximum or minimum—consider an odd-degree polynomial, for instance. However, if you narrow your focus…

The Extreme Value Theorem: The values of a continuous function, from a closed interval of the domain, achieve both a maximum and a minimum value. If you look at a small portion of a function, then in that area, you're *guaranteed* to get a maximum and a minimum.

Local (Relative) Extrema

A local (or relative) maximum is a value that is larger than other functions values for some (small) interval of the domain. Technically: $f(c)$ is a local maximum (value) of $f(x)$ if there exists some open interval (a,b) so that c is in the interval, and $f(c) \geq f(x)$ for all x in the interval (a,b).

Again, there are similar definitions and technical details about local minima (plural of minimum).

Think about the graph of a function with a maximum value…what must be happening to the function around the extreme point? Specifically, what's got to be happening with the slope of the function?

If $f(x)$ has a local maximum (or minimum) of $f(c)$, and $f'(c)$ exists, then $f'(c) = 0$.

Remember your logic—that statement *does not* mean that at every point where the derivative equals zero, we get an extreme value. That's the inverse of the statement, and the inverse of a statement does not have the same truth value as the original statement.

The contrapositive of a statement does have the same value—if $f'(c) \neq 0$, then $f(c)$ is not an extreme value.

Finding Extrema

So, to find extreme values, we should look for points where the first derivative is equal to zero. If there's an extreme value, we'll find it there.

Read carefully—the derivative must exist. Thus, we must also check those points where the derivative does not exist. There are two ways where the derivative fails to exist, even though the function does exist: a vertical tangent (look at the cube root graph near the origin), or a cusp/corner (the vertex of the absolute value graph).

The collection of x-values where either $f'(x)=0$ or $f'(x)$ fails to exist are called the **critical points** of f. We use these points as candidates for local extrema.

To determine if one of your candidates is actually the location of an extremum, then (for now) check the function values immediately around the candidate.

If you've been told to focus your attention on a closed interval, then chances are you've been asked for absolute extrema on that interval. The candidates are the critical values *and* the endpoints of the interval. To determine which candidates are absolute extrema, just compare the function values at each of the candidates.

If you've been asked for local extrema on a closed interval (rare; unusual), then look at the function values around each of the candidates—though, or course, you can only look at one side of each interval endpoint!

Examples

[1.] Find the critical values of $g(t)=\sqrt{t}(1-t)$.

First, the derivative: $g'(t)=\sqrt{t}(-1)+\frac{1}{2\sqrt{t}}(1-t)$. Now, where is this equal to zero?

$$-\sqrt{t}+\frac{1}{2\sqrt{t}}(1-t)=0\Rightarrow\frac{1}{2\sqrt{t}}(1-t)=\sqrt{t}\Rightarrow 1-t=2t\Rightarrow\frac{1}{3}=t$$

It is simple to see that the derivative is undefined at $t=0$. It's also undefined for $t<0$, but those values aren't in the domain of $g(t)$, so we won't worry about them.

The critical values of $g(t)$ are 0 and $\frac{1}{3}$.

[2.] Find the absolute extrema of $f(x)=x^3-3x+1$ on the interval $[0,3]$.

Let's round up some candidates—critical values first. $f'(x)=3x^2-3$; this equals zero when $x=\pm1$. The derivative is defined for all real numbers, so the only critical values are 1 and -1. However, we're only considering points in the interval $[0,3]$, so we only need consider $x=1$.

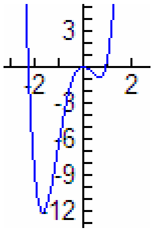
Don't forget to add the endpoints to the list!

The potential absolute extrema are $(0,1)$, $(1,-1)$, and $(3,19)$. The absolute maximum of 19 is at $x=3$, and the absolute minimum of -1 is at $x=1$.

[3.] Find all local extrema of $s(t)=3t^4+4t^3-6t^2$.

The derivative: $s'(t)=12t^3+12t^2-12t$. The critical values: $12t^3+12t^2-12t=0\Rightarrow t\left(t^2+t-1\right)=0$. We get one value easily: $t=0$. However, $t^2+t-1=0$ doesn't factor. It does have roots! Specifically, $t=\frac{-1\pm\sqrt{5}}{2}$. As the domain of the derivative is all real numbers, these three values are all of the critical values.

To determine which candidates are extrema, let's look at the graph.



We get local minima at $t = \frac{-1 \pm \sqrt{5}}{2}$, and a local maximum at $t = 0$.

4.2: *Mean Value Theorem*

The Theorem

Let $f(x)$ be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). There will be some *x*-value $c \in (a,b)$ so that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

In other words, if the initial conditions are met, then there will be a point where the slope of the tangent line (instantaneous rate of change) equals the slope of the secant line (average rate of change).

Increasing and Decreasing

$f(x)$ is **increasing** on the interval (a,b) if $f'(x) > 0$ for every *x*-value in the interval.

$f(x)$ is **decreasing** on the interval (a,b) if $f'(x) < 0$ for every *x*-value in the interval.

Other Ideas

$f(x)$ is **constant** on the interval (a,b) if $f'(x) = 0$ for every *x*-value in the interval.

Let's take this in an unexpected direction: suppose that we have two functions with identical derivatives—$g'(x) = h'(x)$. As a result, we know that $g'(x) - h'(x) = 0$. However, we can consider $g'(x) - h'(x)$ to be a single derivative: if $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) = h'(x)$. Thus, we've got a function whose derivative is equal to zero—thus, the function is constant: $f(x) = k$. Since $f(x) = g(x) - h(x)$, we get $g(x) - h(x) = k \Rightarrow g(x) = h(x) + k$.

The result? If we have functions with identical derivatives, then the functions must differ by a constant. Don't forget that we already know that the inverse of this is also true! This will be very important later on…

One more idea: rather than starting with $f(x)$ and finding $f'(x)$, why don't we start with $f'(x)$ and find $f(x)$?

If you are given a function $g(x)$, then the **antiderivative** of $g(x)$ is a function $f(x)$ so that $f'(x)=g(x)$. You can use the first idea to determine the number of antiderivatives of any function…

Examples

[4.] Find the intervals on which the function $f(x)=x^4-6x^2$ is increasing and decreasing.

First, the derivative: $f'(x)=4x^3-12x$. Now, let's determine where this function is positive and negative: $4x^3-12x=0 \Rightarrow 4x\left(x^2-3\right)=0$; so the points where it *can* change sign are $\left\{-\sqrt{3},0,\sqrt{3}\right\}$. Make a sign chart (I won't try to replicate it here) to see that the function is decreasing (the derivative is negative) on $\left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$; and the function is increasing (the derivative is positive) on $\left(-\sqrt{3},0\right)\cup\left(\sqrt{3},\infty\right)$.

[5.] Find the value of x which satisfies the Mean Value Theorem for the function $g(x)=x^3+x-1$ on the interval $[0,2]$.

The endpoints of the interval are $(0,-1)$ and $(2,9)$, so the slope of the secant line is 5. Where is the derivative equal to five? $g'(x)=3x^2+1$; $3x^2+1=5 \Rightarrow 3x^2=4 \Rightarrow x=\pm\sqrt{\frac{4}{3}}$. The value that is in the given interval is $x=\sqrt{\frac{4}{3}}$.

4.3: Connecting f' and f'' with the Graph of f

The First Derivative Test

In the immediate vicinity of a maximum, what happens to the slope of the curve? It must be positive (rising curve), then negative (falling curve)…

Now think about the slope of the curve around a minimum…

The First Derivative Test: To determine if an extremum candidate is in fact an extremum, look at the sign of the first derivative. Positive followed by negative makes a maximum; the reverse makes a minimum. No change in sign indicates that the function just levels off for a moment.

Concavity

Few functions are constant—most rise and fall. As a result, the derivative of the function is positive and negative.

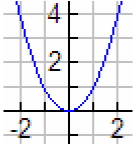
Must the derivative be constant? Must the slope of the curve be the same?

Of course not! Thus, we can ask ourselves about how the derivative rises and falls—we can think about the rate of change of the derivative, and its effect on the graph of *f*.

The first derivative gives information about the slope of the curve—about how the curve rises and falls. The second derivative gives information about how the graph curves. We call this **concavity**.

The graph of $f(x)$ is **concave up** where $f''(x) > 0$, and **concave down** where $f''(x) < 0$.

Think about what this must look like graphically: if $f''(x) > 0$, then the rate of change of the slope is positive. That means that as we move from left to right, the slopes of the curve must be getting bigger. How will the graph curve in order to make this true?



(*how will the graph curve if it is concave down?*)

A point on the graph where the slope (first derivative) changes between positive and negative is an extremum. A point where the concavity (second derivative) changes between positive (up) and negative (down) has a name, also—this is called a **point of inflection**. At a point of inflection, the second derivative must equal zero. Note that where the second derivative is zero, there does not have to be a point of inflection!

The Second Derivative Test

Studying concavity gives us another way to determine if an extremum candidate is actually an extreme value…provided our candidate is the type where the first derivative is equal to zero.

If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a minimum.

If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a maximum.

If $f'(c) = 0$ and $f''(c) = 0$, then we're stuck—we'll have to rely on other methods to determine if we've got an extremum…

Finding f from f'

We've spent quite a bit of time thinking about how f affects f'—but we also need to go backwards! If you are given a graph of f', you need to be able to get some idea about how f looks. Fortunately, we mentioned this in the previous chapter, so it's nothing new…

Examples

[6.] Determine the intervals of concavity, and the points of inflection, of $y = (x^2 - 1)^3$.

For that, we need the second derivative: $y' = 3(x^2 - 1)^2(2x)$, and

$y'' = (6x)(2)(x^2 - 1) + (x^2 - 1)^2(6)$. Perhaps we should clean that up a bit:

$y'' = (6)(x^2 - 1)(2x + (x^2 - 1))$. Now—where is that equal to zero? The first two come out easily: ±1. The remaining two require a calculator, or the quadratic formula: $-1 \pm \sqrt{2}$. A sign chart will show that the function is concave down (has a negative second derivative) when

$x \in \left(-1,-1-\sqrt{2}\right)\cup\left(-1+\sqrt{2},1\right)$; and that the function is concave up (has a positive second derivative) when $x \in \left(-\infty,-1\right)\cup\left(-1-\sqrt{2},-1+\sqrt{2}\right)\cup\left(1,\infty\right)$.

The function has points of inflection at $x=-1$, $x=-1-\sqrt{2}$, $x=-1+\sqrt{2}$, and $x=1$ (since the second derivative changes sign at each of these values).

[7.] Use the second derivative test to locate all local extrema of $h\left(x\right)=x^5-5x+3$.

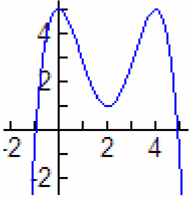
The candidates are determined by the first derivative: $h'\left(x\right)=5x^4-5$. This is equal to zero at $x=\pm1$. We hope to check these candidates with the second derivative test. $h''\left(x\right)=20x^3$. Since $h''\left(-1\right)<0$, there is a maximum at $x=-1$. Since $h''\left(1\right)>0$, there is a minimum at $x=1$.

[8.] Sketch a possible graph of $f\left(x\right)$ if $f'\left(0\right)=0$, $f'\left(2\right)=0$, $f'\left(4\right)=0$, $f'\left(x\right)>0$ for $x\in\left(-\infty,0\right)\cup\left(2,4\right)$, $f'\left(x\right)<0$ for $x\in\left(0,2\right)\cup\left(4,\infty\right)$, $f''\left(x\right)>0$ for $x\in\left(1,3\right)$, and $f''\left(x\right)<0$ for $x\in\left(-\infty,1\right)\cup\left(3,\infty\right)$.

Focus! A chart may be useful.

Interval	$\left(-\infty,0\right)$	$\left(0,1\right)$	$\left(1,2\right)$	$\left(2,3\right)$	$\left(3,4\right)$	$\left(4,\infty\right)$
f' says…	Increasing	Decreasing	Decreasing	Increasing	Increasing	Decreasing
f'' says…	Concave -	Concave -	Concave +	Concave +	Concave -	Concave -

Piece those together…maybe you'll get a graph like this:



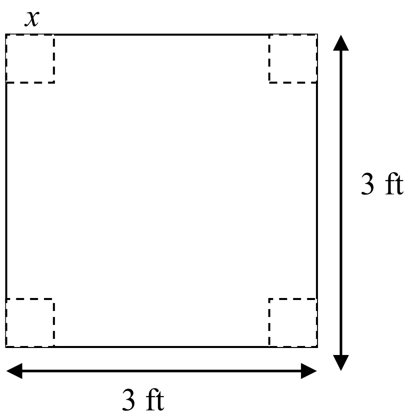
4.4: *Modeling and Optimization*

There are no fancy techniques here—just work! Two pieces of advice: remember to write an equation about the quantity that is to be maximized/minimized, and use the given information to reduce the problem to a single independent variable.

Examples

[9.] A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide on a side, by cutting out a square from each corner and bending up the sides. What is the maximum volume that such a box can have?

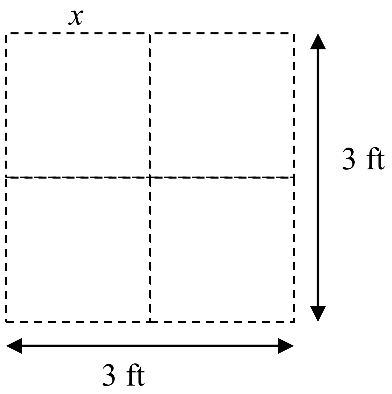
We've been asked to maximize the volume—so we need a volume equation! Perhaps some thought about the situation is required in order to do that…



The base of the box will be a square with side lengths $3-2x$. The length of the square cut will become the height of the box—*x*. The volume of the box will be $V(x)=(3-2x)^2 x$. To identify the maximum value of this function, we need to take find the derivative, and the critical numbers.

$V'(x)=2(3-2x)(-2)x+(3-2x)^2=(3-2x)(-4x+3-2x)=(3-2x)(3-6x)$. This equals zero when $x=\frac{3}{2}$ or $x=\frac{1}{2}$. Note that since the original 3 was in feet, *x* is in feet also.

What would the box look like if $x=\frac{3}{2}$?



Well…that doesn't leave much to fold, does it? The only real candidate for a solution is $x=\frac{1}{2}$. To check that this is, in fact, an extremum, let's use the second derivative test.

$V''(x)=(3-2x)(-6)+(-2)(3-6x)$;

$$V''\left(\frac{1}{2}\right)=\left(3-2\cdot\frac{1}{2}\right)(-6)+(-2)\left(3-6\cdot\frac{1}{2}\right)=(2)(-6)+(-2)(0)<0$$

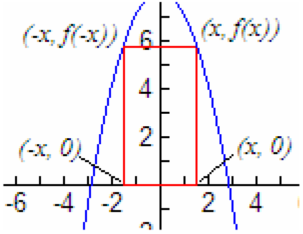
Since the second derivative is less than zero, there is a maximum at $x=\frac{1}{2}$.

This doesn't answer the question! The maximum volume is

$$V\left(\frac{1}{2}\right)=\left(3-2\cdot\frac{1}{2}\right)^2\left(\frac{1}{2}\right)=(2)^2\left(\frac{1}{2}\right)=2.$$

[10.] Find the dimensions of the largest rectangle with the base vertices on the x-axis and the other vertices on the graph $y = 8 - x^2$.

This time, we need to maximize the area of the rectangle. Let's think about it…



The base of the rectangle is 2x, and the height is $8 - x^2$; the area is $A(x) = (2x)(8 - x^2) = 16x - 2x^3$.

Derivative time! $A'(x) = 16 - 6x^2$

Critical points time! $16 - 6x^2 = 0 \Rightarrow 16 = 6x^2 \Rightarrow \frac{8}{3} = x^2 \Rightarrow \pm\sqrt{\frac{8}{3}} = x$

(do we really need both the positive and negative values? No!)

Second derivative time! $A''(x) = -12x$; $A''\left(\sqrt{\frac{8}{3}}\right) = -12\sqrt{\frac{8}{3}} < 0$.

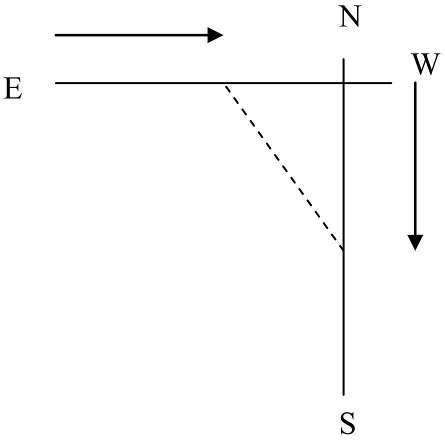
Thus, the base of the rectangle should be $2x = 2\sqrt{\frac{8}{3}}$, and the height should be

$$8 - x^2 = 8 - \left(\sqrt{\frac{8}{3}}\right)^2 = 8 - \frac{8}{3} = \frac{16}{3}.$$

[11.] A boat leaves a dock at 2 pm and travels due south at 20 km per hour. Another boat, headed due west at 15 km per hour, arrives at the same dock at 3 pm. At what time were the boats closest together?

At least it's not a train leaving Chicago…

Note that *closest together* means *minimum distance*. We need a distance formula for these boats. Time for a diagram.



The dashed line is the distance between the boats at some time. Notice that this is the hypotenuse of a right triangle! Now—if we just knew the lengths of the other two sides…

The North-South line is easy: $20t$, where t is in hours.

What about the East-West line? It isn't the distance traveled by that boat; it's the distance remaining before it reaches the dock. Hmmm.

Well, that boat is going 15 km per hour, and it reaches the dock one hour after the other boat departs…when the N-S boat sets off, the E-W boat must be 15 km from the dock! Thus, the distance remaining must be $15-15t$.

That makes the distance between them $D(t)=\sqrt{(20t)^2+(15-15t)^2}$.

The derivative: $D'(t)=\dfrac{1}{2\sqrt{(20t)^2+(15-15t)^2}}\left[2(20t)(20)+2(15-15t)(-15)\right]$.

This needs to be cleaned up!

$$D'(t)=\frac{400t-225+225t}{\sqrt{(20t)^2+(15-15t)^2}}=\frac{625t-225}{\sqrt{(20t)^2+(15-15t)^2}}$$

The derivative equals zero when $625t=225\Rightarrow t=\dfrac{9}{25}$.

I won't use the second derivative test here—instead, consider the sign of $D'(t)$ to either side of $t=\dfrac{9}{25}$. Note that the denominator must be positive; the numerator switches from negative to positive. Thus, at $t=\dfrac{9}{25}$, the boats are closest together.

4.5: Linearization and Newton's Method

Linear Approximation

We've defined the slope of a curve at a point to be the slope of the line tangent to the curve at that point…what we haven't pointed out is that if you look really, really closely, around that point, the tangent line is virtually indistinguishable from the curve. Thus, the tangent line can be used as a **linear approximation of the curve** around that point.

Consider a function $f(x)$ around the point $x=a$. The point on the curve (and the tangent line) is $(a,f(a))$, and the slope of the tangent line is $f'(a)$. Thus, the equation of the tangent line is $y-f(a)=f'(a)(x-a)\Rightarrow y=f(a)+f'(a)(x-a)$. This serves as a linear approximation of $f(x)$ for values of *x* close to *a*.

Sometimes, we're not given a specific value of *x*, but a small increment from *a*—call it *h*. That makes $x=a+h$, and the linear approximation $y=f(a)+f'(a)([a+h]-a)=f(a)+f'(a)(h)$.

Summarizing:

$$f(x)\approx f(a)+f'(a)(x-a)$$
$$f(a+h)\approx f(a)+f'(a)(h)$$

Differentials

We know that $\frac{dy}{dx} = f'(x)$…but the stuff on the left side of that equation is a little mysterious. They're called **differentials**—especially if you rewrite the previous equation as $dy = f'(x)dx$. The differential dy is the change that occurs in y ($f(x)$) for a small change in x (dx). This is exactly what we just did with the previous linear approximation!

Rewrite: $f(a+h) - f(a) \approx f'(a)(h)$. The left side is really dy, and the h is really dx.

Examples

[12.] Find the linear approximation of $y = \sqrt{1-x}$ at $x = 0$. Use this linearization to approximate the value of $\sqrt{0.9}$.

So $f(x) = \sqrt{1-x}$, $a = 0$, and $h = 0.1$.

The derivative: $y' = \frac{-1}{2\sqrt{1-x}}$. $f'(a) = f'(0) = \frac{-1}{2\sqrt{1-0}} = \frac{-1}{2}$. $f(a) = f(0) = \sqrt{1} = 1$. The tangent line approximation is $(1) + \left(-\frac{1}{2}\right)(0.1) = 1 - 0.05 = 0.95$.

[13.] A circular disk has a measured radius of 24 cm. Use a linear approximation to determine the maximum error in the calculation of the area of the disk if the maximum error in the measurement of the radius is 0.2 cm.

First of all, the equation that we're approximating is area: $A(r) = \pi r^2$. The fixed point (a) is 24, and the increment (h) is 0.2. Really, the error could be larger or smaller, but think about our function…it's a parabola. It rises faster to the right than to the left; we need only consider the right-hand case.

$A'(r) = 2\pi r$; $A(24) = 576\pi$; $A'(24) = 48\pi$. The tangent line approximation is $576\pi + (48\pi)(0.2) = 576\pi + 9.6\pi = 585.6\pi$.

That's just the area; the error is the difference from the original area…which is right there in the linear approximation! The estimate of error is $9.6\pi \approx 30.159$ (the units are square centimeters).

4.6: Related Rates

Take these step by step, and they will be a lot easier than anyone else thinks they are!

First, encode the question in terms of variables—write the given rates as derivatives, write the given values as variables.

Next, write an equation that relates your variables. Make sure that the derivative of this equation (almost always implicit) will result in the derivatives (rates) that were given/requested in the statement of the problem.

Now—and only now—plug in the values of those rates and variables. This should leave you with just one unknown quantity!

Now for some examples…

Examples

[14.] Air is being pumped into a spherical balloon at the rate of $100\frac{cm^3}{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

As you read this, take notes—assign values to variables and derivatives.

at the rate of $100\frac{cm^3}{s}$…so $\frac{dV}{dt}=100$.

How fast is the radius of the balloon increasing…what is $\frac{dr}{dt}$?

when the diameter is 50 cm…when $r=25$.

We need to write an equation that, when differentiated, produces $\frac{dV}{dt}$ and $\frac{dr}{dt}$—thus, we need an equation that uses *V* and *r*.

How about volume? The volume of a sphere is $V=\frac{4}{3}\pi r^3$.

Don't plug in that value of *r* just yet! I know it's tempting, but it will ruin everything. If a number is in there, instead of a variable, then the derivative won't include $\frac{dr}{dt}$.

Okay—take the derivative with respect to time. $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$.

Now—plug in. $100=4\pi\left(25\right)^2\frac{dr}{dt}$.

Finally, solve. $100=4\pi\left(25\right)^2\frac{dr}{dt}\Rightarrow\frac{100}{4\pi\left(625\right)}=\frac{dr}{dt}=\frac{1}{25\pi}\approx 0.013$. The radius is increasing at 0.013 centimeters per second.

[15.] A certain water tank is the shape of an inverted cone, with base radius 2 m and height 4 m. Water is being pumped into the tank at the rate of $2\frac{m^3}{\min}$. How fast is the water level in the tank rising when the water is 3 m deep?

at the rate of $2\frac{m^3}{\min}$… $\frac{dV}{dt}=2$.

How fast is the water level in the tank rising…what is $\frac{dh}{dt}$?

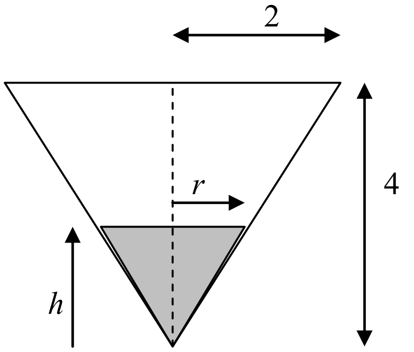
when the water is 3 m deep…when $h=3$.

Once again, a volume equation is needed: $V=\frac{1}{3}\pi r^2 h$.

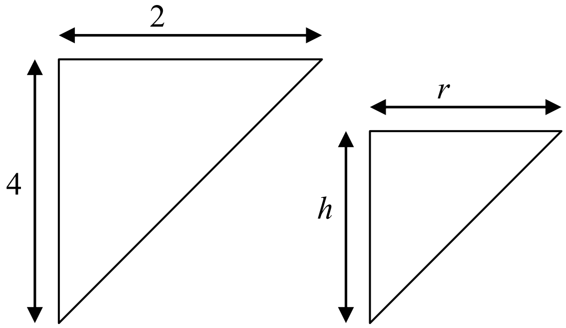
Stop! Think! If you take the derivative of that equation as it is right now, what's going to happen?

Product rule! And, you'll end up with $\frac{dr}{dt}$…something that wasn't given in the information, and wasn't requested as an answer.

You've got too many variables! The question is how to eliminate one…and the answer is surprisingly simple. Let's look at a picture of this cone.



Is there a relationship between h and r? Yes!



These are similar triangles—so $\frac{h}{4}=\frac{r}{2}\Rightarrow\frac{h}{2}=r$.

This makes the volume equation $V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi}{12}h^3$.

Now take the derivative: $\frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}$.

Next, plug in: $2=\frac{\pi}{4}(3)^2\frac{dh}{dt}$.

Solve! $2=\frac{\pi}{4}(3)^2\frac{dh}{dt}\Rightarrow\frac{8}{9\pi}=\frac{dh}{dt}\approx 0.283$. The height is increasing at 0.283 meters per second.

[16.] Stranded on Tatooine, R2-D2 and C-3PO have decided to split up in the desert. R2-D2 moves due north at 7km/hr, and C-3PO moves due east at 5km/hr. How fast is the distance between them increasing after 2 hours?

Let's let n = distance north (of R2-D2), e = distance east (of C-3PO), and D = distance between the two. So we're given $\frac{dn}{dt}=7$, $\frac{de}{dt}=5$, and $t = 2$.

The only equation we can write that relates these is $n^2 + e^2 = D^2$. Of course, that means $2n\frac{dn}{dt}+2e\frac{de}{dt}=2D\frac{dD}{dt}$. What are we going to plug in for n, e and D? How will we use the value $t = 2$?

Of course! At $t = 2$, $n = 14$, $e = 10$, and $D=\sqrt{14^2+10^2}=\sqrt{196+100}=\sqrt{296}$.

$2(14)(7)+2(10)(5)=2\sqrt{296}\,\dfrac{dD}{dt}$, so $\dfrac{dD}{dt}=\dfrac{296}{2\sqrt{296}}=\dfrac{148}{\sqrt{296}}$, or approximately 8.602 kilometers per hour.