3: Derivatives

3.1: The Derivative of a Function

The Slope Function

So far, we've talked about finding the slope of a curve at a point. However, during some of those problems, we pretended that we didn't know the point—we put another variable in that spot. Thus, the process of finding a slope resulted in a *function*, rather than a *number*. This is an important result!

The **derivative** of a function $f(x)$ is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The derivative can be thought of as a *slope function*. $f(x)$ gives the *y-value of the graph* for a given *x*, and $f'(x)$ gives the *slope of the graph* for a given *x*.

Since this definition requires a limit, it only works if the limit from the left equals the limit from the right. Thus, there will be times when we have a derivative on the left and a different derivative on the right—which means that there is no derivative at that point.

If you just want the value of the derivative at a point, then you can (and maybe you should) use your calculator. Under the MATH menu is the command `nDeriv(`.This stands for *numeric derivative*. Give it a function, tell it which variable is independent, and give it a value for that variable, and it will grind through some calculations to find the value of the derivative at that point—the slope of the function at that point.

There are many notations for the derivative of a function, but they fall into two basic camps: the "prime notation," and Leibniz's notation.

Prime notation: y' or $f'(x)$.

Leibniz's notation: $\frac{dy}{dx}$, $\frac{d}{dx}\left[f(x)\right]$.

Each has its uses—you must be familiar with both.

There will be occasions where you don't actually know the function—where you only have a few points. From these, you can form secant lines, and you should be able to get some idea of what the tangent lines look like.

The Relationship Between a Function and its Derivative

It is important to think about how the graph of the function will affect the graph of its derivative, and vice versa. Some important items to remember:

- Where the graph is flat, the derivative is zero.
- Where the graph rises, the derivative must be positive.
- Where the graph falls, the derivative must be negative.

Examples

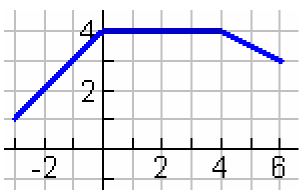
[1.] If $g(x)=\sqrt{x}$, then find $g'(x)$.

Whew! Here goes. $g'(x)=\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$. Watch this trick to work on this:

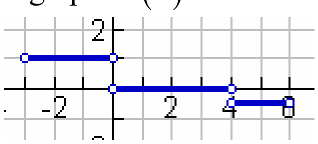
$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{(x+h)-(x)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

At this point, it is okay to let h be zero! $g'(x)=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}}=\dfrac{1}{\sqrt{x}+\sqrt{x}}=\dfrac{1}{2\sqrt{x}}$.

[2.] Use the graph of $h(x)$ below to sketch the graph of $h'(x)$.

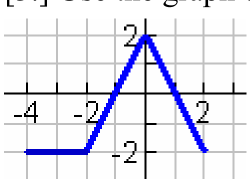


So, I need to find the slope of the graph at each value of x. On the interval $[-3,0]$, the slope is 1. On the interval $[0,4]$, the slope is zero. On the interval $[4,6]$, the slope is $-\dfrac{1}{2}$. Thus, I can now graph $h'(x)$:



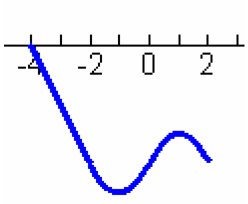
(we'll talk about why those are open circles in the next section)

[3.] Use the graph of $j'(x)$ below to sketch the graph of $j(x)$.



The slope of the function is constant at -2 for x-values between -4 and -2—that will make the function graph a line with slope -2. Then, the slope begins changing…it is still negative for x-values between -2 and -1, but it is getting closer to zero. This means that the function is going down, but in a curve, so that it flattens out at -1. The slopes are positive after that, and getting higher—again, a curve. At an x-value of zero, the *slope* (not the function) reaches a maximum value of two, and then begins heading back to a slope of zero. Think about that again: the slope is still positive, but as you move from left to right, the slopes are flattening back towards zero. When x is 1, the slope is zero, and then curves down.

Here's a possible answer:



(there are actually an infinite number of answers…more on that later)

3.2: Differentiability

Points Without Slopes

I've already mentioned one case where the derivative fails to exist at a point—when the left and right hand derivatives don't come out to the same the value. Graphically, this looks like a sharp point on the graph; a sudden change from one slope to another, without taking any slope values between the two. The classic example of this is the absolute value function at $x = 0$.

The textbook actually divides this problem into three cases: *corner*, *cusp* and *discontinuity*. However, in all cases, the left hand derivative comes out to a different value than the right hand derivative.

The other instance where a derivative fails to exist is where the slope approaches infinity—graphically, a *vertical tangent*. This is best seen in the graph of $y = \sqrt[3]{x}$ at $x = 0$.

Local Linearity

Take a graph—any graph. Pick a point on that graph. Now, zoom in. Zoom in *a lot*. What do you see?

It should look like a line!

If you zoom in enough on a graph, then the graph starts to look like its tangent line. On very small intervals, the tangent line at a point is virtually indistinguishable from the function. On very small intervals, a line can be used to get approximate values of any function—you just need to know a bit about the function at a point in that interval.

Let's generalize: the slope of $f(x)$ at the point $(a, f(a))$ is $f'(a)$. Thus, the equation of the tangent line at that point is $y - f(a) = f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a)$. On a very small interval, *y* (from the tangent line) and $f(x)$ are virtually indistinguishable, so $f(x) \approx f(a) + f'(a)(x-a)$. If you want to approximate the value of the function just a tiny bit away from the point $(a, f(a))$—say, at $x = a + h$—then we can say that $f(a+h) \approx f(a) + f'(a)((a+h)-a) = f(a) + f'(a)(h)$.

Consequences of the Derivative

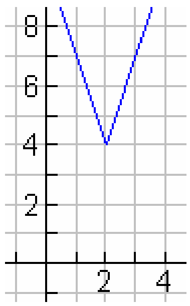
If the derivative exists, then the function it came from must be continuous—otherwise, the limit definition of the derivative wouldn't have come out. Thus, differentiability implies continuity.

Note that continuity does not imply differentiability—consider the absolute value function. It is continuous on the real numbers, but has no derivative at zero.

We already know that a function $f(x)$ which is continuous on the interval $[a,b]$ must take on every value between $f(a)$ and $f(b)$. As a bonus, if $f(x)$ is differentiable on $[a,b]$, then $f'(x)$ takes on every value between $f'(a)$ and $f'(b)$.

Examples

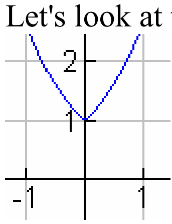
[4.] Compute the left and right hand derivatives to show that $g(x)$ is not differentiable at $x=2$.



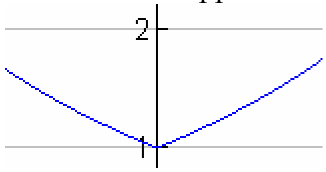
The derivative on the left—the slope of the graph to the left of $x=2$ is -3; the derivative on the right is 3. Since the left and right hand derivatives are not equal, the function does not have a derivative (there is no tangent line) at $x=2$.

[5.] Determine all values of x for which $h(x)=e^{|x|}$ is differentiable.

Let's look at the graph.



This function appears to have a problem at $x=0$, Let's zoom in.



The left and right hand slopes don't appear to smoothly transition there…this function is differentiable for $x \in (-\infty,0)\cup(0,\infty)$.

[6.] Find the slope of $y=\ln|x|-x^2$ at $x=\frac{1}{2}$.

The sheer nastiness of that function should have you running to the calculator at once! It reports the slope to be 1 at that point.

3.3: Rules for Differentiation

Constants

A constant function is of the form $y = k$, where $k \in \mathbb{R}$. Graphically, it's a flat line. Thus, what's the slope? Zero, of course!

$$\frac{d}{dx}[k] = 0$$

Power Functions

A power function is of the form $y = x^n$, where *n* is some kind of number…for now, we'll assume that it is an integer other than zero ($n \in \mathbb{Z} - \{0\}$).

From the derivatives we've already computed, you probably have an idea of the derivative of a power function. We will prove the following result in class:

$$\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$$

Constant Multiples

A constant multiple is a known function (power, exponential, etc.) multiplied by a real number: $2x^3$, $-5 \cdot \cos(x)$, etc. It should be a simple matter to realize that *these* constants get factored out of the limiting process that makes a derivative. Thus:

$$\frac{d}{dx}\left[k \cdot f(x)\right] = k \cdot f'(x)$$

Sums and Differences

Again, simple…and so useful!

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

Products

We will not *rigorously* prove this…the hard part for you will be recognizing when it is appropriate to use this rule.

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotients

No proof. Realizing *when* to use it, and *remembering how* to use it will be sufficiently strenuous.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

Higher Order Derivatives

What we've been finding so far is only the first derivative. Of course, you can take the derivative of the derivative, resulting in the second derivative. And then, you can do it again…

The notations for the second derivative are (basically) y'' or $\frac{d^2y}{dx^2}$.

The notations for higher derivatives—the n[th] derivative—are $y^{(n)}$ and $\frac{d^ny}{dx^n}$.

Examples

[7.] Find y' if $y = 4x^2 - 5x + 7$.

Using the first four rules, we get $y' = 8x - 5$.

[8.] Find $\frac{dy}{dx}$ if $y = \left(2x^9 + 7x^5 - x^3 + 1\right)\left(5x^7 - 7x^4 - 6x^2 - 17\right)$.

Now, if you really, *really* want to, you can expand that 16[th] degree polynomial and avoid the product rule entirely…however, I'm not going to wait while you do that. I'm going to use the product rule.

$$\frac{dy}{dx} = \left(18x^7 + 35x^4 - 3x^2\right)\left(5x^7 - 7x^4 - 6x^2 - 17\right) + \left(2x^9 + 7x^5 - x^3 + 1\right)\left(35x^6 - 18x^3 - 12x\right)$$

[9.] Find $\frac{d}{dx}\left[\frac{2x+1}{x-2}\right]$.

Unless you want to do some polynomial division, you're going to have to use the quotient rule.

$$\frac{d}{dx}\left[\frac{2x+1}{x-2}\right] = \frac{(2)(x-2)-(2x+1)(1)}{(x-2)^2}$$

(*you don't feel any need to simplify that, do you?*)

[10.] Find $f''(x)$ if $f(x) = 4x^5 + 7x^3 - 6x + 10$.

$f'(x) = 20x^4 + 21x^2 - 6$, so $f''(x) = 80x^3 + 42x$.

3.4: Velocity and Other Rates of Change

We motivated the idea of derivatives through instantaneous speed, so there's no need to rehash the average/instantaneous stuff. Note, however, that all of the following items are *instantaneous*, and are for motion along a line (back and forth, or up and down). Problems relating to these definitions are called **particle motion** problems.

If $f(t)$ represents the **position** (distance, displacement) of an object at any time *t*, then $f'(t)$ gives the **velocity** of the object, and $f''(t)$ gives the **acceleration**.

To resolve the speed/velocity problem, here's a definition: the **instantaneous speed** of an object is $|v(t)| = |f'(t)|$.

Here are some useful formulas for objects undergoing constant acceleration—say, a falling object? Note that a falling object has a *negative* acceleration.

$$a(t) = g \; ; \; v(t) = gt + v_0 \; ; \; h(t) = \frac{1}{2}gt^2 + v_0 t + h_0$$

In economics, the term marginal refers to the derivative—marginal revenue is the derivative of revenue, etc.

In all word problems, keep track of units!

Examples

[11.] A particle moves along the *x*-axis so that its position (*x*-coordinate) at any time $t \in [0,10]$ seconds is given by $f(t) = t^3 - 12t^2 + 36t$. When is the particle at rest?

Being at rest means a velocity of zero—the equation for velocity is the derivative of position: $v(t) = f'(t) = 3t^2 - 24t + 36$. So, when is this zero?

$$3t^2 - 24t + 36 = 0 \Rightarrow t^2 - 8t + 12 = 0 \Rightarrow (t-6)(t-2) = 0$$

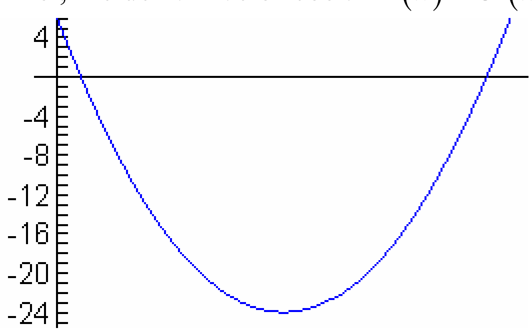
So, the particle is at rest at 2 seconds and 6 seconds.

[12.]A particle moves along the *x*-axis so that its position (*x*-coordinate) at any time $t \in [0,6]$ seconds is given by $f(t) = t^3 - 9t^2 + 15t + 10$. When is the acceleration of the particle negative?

Derivative time! $f'(t) = 3t^2 - 18t + 15$, and $f''(t) = 6t - 18$. $6t - 18 < 0 \Rightarrow 6t < 18 \Rightarrow t < 3$. The acceleration is negative for times between 0 and 3 seconds.

[13.] The cost (in dollars) to produce *x* items of a certain product is $C(x) = 0.001x^3 - 0.3x^2 + 6x + 900$. At what production level does the marginal cost of producing this product begin to increase?

First, the derivative of cost: $M(x) = C'(x) = 0.003x^2 - 0.6x + 6$. Let's look at the graph:



A quick zoom will determine that this begins increasing at $x = 100$ units.

3.5: Derivatives of Trigonometric Functions

Derivatives of Sine and Cosine

We'll prove these in class.

$\frac{d}{dx}\left[\sin(x)\right]=\cos(x)$

$\frac{d}{dx}\left[\cos(x)\right]=-\sin(x)$

"Jerk"

The text defines a sudden change in acceleration to be a "jerk." The change in acceleration is given by the derivative of acceleration…acceleration is the second derivative of position, so "jerk" can be measured with the third derivative of position.

Wheee!

Derivatives of Other Trigonometric Functions

Again, we'll prove these in class.

$\frac{d}{dx}\left[\tan(x)\right]=\sec^2(x)$

$\frac{d}{dx}\left[\cot(x)\right]=-\csc^2(x)$

$\frac{d}{dx}\left[\sec(x)\right]=\sec(x)\cdot\tan(x)$

$\frac{d}{dx}\left[\csc(x)\right]=-\csc(x)\cdot\cot(x)$

Examples

[14.] Find $f'(x)$ if $f(x)=x\cdot\cos(x)$.

Product rule, anyone? $f'(x)=(1)(\cos(x))+(x)(-\sin(x))=\cos(x)-x\cdot\sin(x)$

[15.] Find the equation of the line tangent to $y=\cot(x)$ when $x=\frac{\pi}{4}$.

First, the equation for the slope of the tangent: $y'=-\csc^2(x)$. Now let $x=\frac{\pi}{4}$ to find the actual slope: $y'=-\csc^2\left(\frac{\pi}{4}\right)=-\sqrt{2}$. Next, the point on the graph: $y=\cot\left(\frac{\pi}{4}\right)=1$. Finally, the equation of the tangent: $y-1=-\sqrt{2}\left(x-\frac{\pi}{4}\right)$.

3.6: The Chain Rule

There is probably no more important rule when it comes to taking derivatives!

The chain rule tells how to take the derivative of a composition of functions.

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

In words, you take the derivative of the outer function, leaving the inner function alone, then multiply by the derivative of the inner function.

Here's another way to think of it: $\frac{d\left(f\right)}{dx}=\frac{d\left(f\right)}{d\left(g\right)}\cdot\frac{d\left(g\right)}{dx}$. This says that we first take the derivative of *f* as if $g\left(x\right)$ was an independent variable, then we take the derivative of $g\left(x\right)$ with *x* as the independent variable.

The chain rule will not go away. Almost every derivative problem will require the use of the chain rule. Train yourself to see the composed functions!

Examples

[16.] Find y' if $y=\left(x^2+3x-2\right)^{24}$.

Hopefully you immediately see that we have one function (x^2+3x-2) inside of another (x^{24}). The derivative of the outer function is $24x^{23}$; plug the inner function into that and multiply by the derivative of the inner function. $y'=24\left(x^2+3x-2\right)^{23}\left(2x+3\right)$.

[17.] Find $\frac{d}{dx}\left[\sqrt{x^3+4x^2-5x-7}\right]$.

Note that the function can be written as $\left(x^3+4x^2-5x-7\right)^{\frac{1}{2}}$. The derivative is

$$\frac{1}{2}\left(x^3+4x^2-5x-7\right)^{-\frac{1}{2}}\left(3x^2+8x-5\right)=\frac{3x^2+8x-5}{2\sqrt{x^3+4x^2-5x-7}}.$$

[18.] Find $f'\left(x\right)$ if $f\left(x\right)=\sin\left(x^3+4x\right)$.

$$f'\left(x\right)=\left[\cos\left(x^3+4x\right)\right]\left[3x^2+4\right]$$

3.7: Implicit Differentiation

An **explicit equation** has one variable isolated, and all other variables on the other side of the equals symbol. You've worked with lots of explicit functions over the years, and that's all we've used so far this year.

An **implicit equation** is not solved for a variable—all of the variables are all mixed up together. You are already familiar with some implicit functions, such as the equation of a circle. What we need to do is learn how to take the derivative of one variable (with respect to another) when the equation is implicitly defined.

There are two tricks: use Leibniz's notation, so that you know which variable is being considered dependent and which is independent; and treat the dependent variable as a function of the independent function.

This is difficult to explain…better to see some examples.

Examples

[19.] Find $\frac{dy}{dx}$ if $x^2 + y^2 = 16$.

It begins simply enough—the derivative of x^2 is easy! However, the next item deserves some attention. The stem of the problem indicates that *y* is the dependent variable—thus, I should treat it as a function. However, note that the term is actually y^2—a function raised to a power. That's a function within a function! I'll have to use the chain rule to take the derivative of y^2.

So, here it is: $2x + 2y\frac{dy}{dx} = 0$. However, the question implies that I need to solve for $\frac{dy}{dx}$:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

[20.] Find $\frac{dv}{du}$ if $u^2 + uv - v^3 = u + v$.

This time, treat *v* as a function of *u*. Also, beware the product rule!

$$2u + \left(v + u\frac{dv}{du}\right) - 3v^2\frac{dv}{du} = 1 + \frac{dv}{du}$$

$$2u + v - 1 = \frac{dv}{du} + 3v^2\frac{dv}{du} - u\frac{dv}{du}$$

$$2u + v - 1 = \left(1 + 3v^2 - u\right)\frac{dv}{du}$$

$$\frac{2u + v - 1}{1 + 3v^2 - u} = \frac{dv}{du}$$

[21.] Find $\frac{d^2y}{dx^2}$ if $x + y = xy$.

Take it one at a time. $1 + \frac{dy}{dx} = y + x\frac{dy}{dx} \Rightarrow 1 - y = x\frac{dy}{dx} - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1-y}{x-1}$.

Now, take the derivative again—since we've solved for $\frac{dy}{dx}$, that will make the left side easy (just what we did up until this section). The right side will result in some more $\frac{dy}{dx}$ terms, which you should substitute for the equation we found just above.

Oh, and…enjoy that quotient rule.

$$\frac{dy}{dx}=\frac{1-y}{x-1}\Rightarrow\frac{d^2y}{dx^2}=\frac{(x-1)\left(-\frac{dy}{dx}\right)-(1-y)}{(x-1)^2}\Rightarrow$$

$$\frac{d^2y}{dx^2}=\frac{(x-1)\left(-\frac{1-y}{x-1}\right)-(1-y)}{(x-1)^2}=\frac{-(1-y)-(1-y)}{(x-1)^2}=\frac{-2-2y}{(x-1)^2}$$

3.8: Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Functions in General

So…you've got an inverse function, and you want to take its derivative. If it's the inverse of a nice polynomial or rational function, no problem—go right ahead. However, other functions will require some…*finesse*.

You've been given $y=f^{-1}(x)$. Let's rearrange things so that the inverse isn't in the way: $f(y)=x$. Now, take the (implicit) derivative as normal: $f'(y)\cdot\frac{dy}{dx}=1\Rightarrow\frac{dy}{dx}=\frac{1}{f'(y)}$.

However, remember that we know what y is—so we can write $\frac{dy}{dx}=\frac{1}{f'\left(f^{-1}(x)\right)}$.

Tada!

Derivatives of Inverse Trigonometric Functions

We'll show why these equations work in class.

$$\frac{d}{dx}\left[\sin^{-1}(x)\right]=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1}(x)\right]=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}(x)\right]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\cot^{-1}(x)\right]=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}(x)\right]=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\csc^{-1}(x)\right]=-\frac{1}{|x|\sqrt{x^2-1}}$$

The text provides some identities for the cosine, cotangent, and cosecant inverse functions—they could come in handy. The big three (the ones you should know) are sine, secant, and tangent.

Examples

[22.] Find $\frac{dy}{dt}$ if $y=\sin^{-1}\left(\sqrt{t}\right)$.

Oooh—chain rule! $\frac{dy}{dt}=\frac{1}{\sqrt{1-\left(\sqrt{t}\right)^2}}\cdot\frac{1}{2\sqrt{t}}=\frac{1}{\sqrt{1-t}}\cdot\frac{1}{2\sqrt{t}}=\frac{1}{2\sqrt{t(1-t)}}$.

[23.] Find $\frac{dx}{d\theta}$ if $x=\sec^{-1}\left(\frac{\pi}{2}-\theta\right)$.

Wheee! $\frac{dx}{d\theta}=\frac{1}{\left|\frac{\pi}{2}-\theta\right|\sqrt{\left(\frac{\pi}{2}-\theta\right)^2-1}}\cdot(-1)$.

[24.] Find $f'(x)$ if $f(x)=\tan^{-1}\left(\frac{1}{x}\right)$.

$f'(x)=\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)=\frac{1}{\frac{x^2+1}{x^2}}\cdot\left(-\frac{1}{x^2}\right)=-\frac{1}{\left(x^2+1\right)}$.

3.9: Derivatives of Exponential and Logarithmic Functions

The Natural Exponential

We'll explore the reasons for this in class…it turns out that $\frac{d}{dx}\left(e^x\right)=e^x$!

Any Exponential

This is harder. We'll start with $y=n^x$. This can be rewritten as $y=e^{\ln\left(n^x\right)}=e^{x\cdot\ln(n)}$. Note that $\ln(n)$ is just a constant; we can handle this with the chain rule.

$\frac{d}{dx}\left(n^x\right)=\frac{d}{dx}\left(e^{x\cdot\ln(n)}\right)=e^{x\cdot\ln(n)}\cdot\ln(n)=n^x\cdot\ln(n)$

The Natural Logarithm

Logarithms are inverse functions of exponentials! Let's use that earlier result: $\frac{dy}{dx}=\frac{1}{f'\left(f^{-1}(x)\right)}$. In this case, $f(x)=e^x$, and $y=f^{-1}(x)=\ln(x)$. So, we get $\frac{dy}{dx}=\frac{1}{e^{\ln(x)}}=\frac{1}{x}$.

Any Logarithm

To take the derivative of any base logarithm, use the change of base formula to turn the logs into natural logs.

It's only natural!

Examples

[25.] Find the equation of the line tangent to $y=\frac{e^x}{x}$ at the point $(1,e)$.

First, the slope equation: $y'=\frac{xe^x-e^x}{x^2}$. Plug in $x=1$ and we get zero! The tangent line is horizontal, at $y=e$.

[26.] 24g of a radioactive substance decays according to the equation $m(t)=24\cdot 2^{-\frac{t}{25}}$, where t is the time in years and the function produces the mass in grams. After 40 years, how fast is the amount of mass decreasing?

The rate of decrease in mass is the derivative of the mass function—the question asks for $m'(40)$. $m'(t)=24\cdot\ln(2)\left(2^{-\frac{t}{25}}\right)\left(-\frac{1}{25}\right)$, so $m'(40)=24\cdot\ln(2)\left(2^{-\frac{40}{25}}\right)\left(-\frac{1}{25}\right)\approx -0.220$. The substance is decreasing at about 0.220 grams per year, after 40 years.

[27.] Find the derivative of $g(x)=\cos(\ln(x))$.

$g'(x)=-\sin(\ln(x))\cdot\frac{1}{x}$.

[28.] Differentiate $y=\log_{10}\left(\frac{x}{x-1}\right)$.

First, convert: $y=\frac{1}{\ln(10)}\cdot\ln\left(\frac{x}{x-1}\right)$. Now, differentiate.

$$y'=\frac{1}{\ln(10)}\cdot\frac{x-1}{x}\cdot\frac{(x-1)-x}{(x-1)^2}=\frac{1}{\ln(10)}\cdot\frac{x-1}{x}\cdot\frac{-1}{(x-1)^2}=-\frac{1}{\ln(10)}\cdot\frac{1}{x(x-1)}.$$