1: Prerequisites for Calculus

This chapter concerns prerequisites…things that you should already know. I won't dwell on them, and I won't give any examples in these notes for this chapter—these are things you should know how to do! These are just a few reminders, some clarifications, technical details, etc.

1.1: Lines

If you don't yet know what a line is, then you're in trouble…ditto and double if you don't know what slope is.

There are three forms for writing the equation of a line with which you should be familiar: **slope-intercept**, **point-slope**, and **general**.

Slope-intercept form writes a line in the form $y = mx + b$, where m is the slope of the line, and b is the y-coordinate of the y-intercept.

Point-slope form writes a line in the form $y - y_1 = m\left(x - x_1\right)$, where m is the slope of the line, and $\left(x_1, y_1\right)$ is any point on the line. Note that the text solves this equation for y; this is not really necessary…

General form (as this text defines it) writes a line in the form $Ax + By = C$, where A and B are not both equal to zero. The values of A and B do not have immediate meanings, but this is the only form that can handle vertical lines (in addition to horizontal and oblique).

You should know that parallel lines have the *same slope* (unless they're both vertical), and that perpendicular lines have *slopes whose product is -1* (unless the lines are horizontal and vertical).

You should be able to write the equation of a line if given two points, or one point and the slope. You should also be able to write the equation of a line that passes through a given point and is parallel or perpendicular to a given line.

1.2: Functions and Graphs

Relations and Functions

A **relation** is any set of points in the plane (that's the simplest way to say that it is a mapping from a set of inputs to a set of outputs). Any equation that you can write automatically defines a relation (though not every relation can be described with an equation).

A **function** is a relation where every input gets mapped to a *single* output—for each value of x, there is *exactly* one value of y. Virtually every function is described with an equation. Graphically, you can tell if a set of points (a graph) is a function by applying the **vertical line test**—if any vertical line intersects the graph in more than one point, then the graph is not a function.

Domain and Range

The **domain** of a relation is the set of x-values that are (can be) used as inputs. We are concerned with *real-valued functions* (we just don't say it often, since it takes so much time…) with equations, so let's say that the domain of a function is that subset of the real numbers (set of x-values) that produces real numbers (y-values).

The **range** of a relation is the set of *y*-values that are outputs. This is much harder to find without some help from the calculator—we'll work on it.

Domain and range can be written in different ways, and you should be able to read (and understand) all of them. The simplest is with full words: "the domain of the function is all real numbers." Here is the same statement using symbols: $x \in \mathbb{R}$. In the most correct (perhaps even obtuse) form, this is read as "*x* is contained in the set of real numbers." Here is one more way to say the same thing: $x \in (-\infty, \infty)$. This is *interval notation*, and you need to become familiar with it (if you aren't already). Here are some examples (domain and range):

Open Interval: $x \in (-2, 4)$. This is equivalent to $-2 < x < 4$.

Closed Interval: $y \in [3, 5]$. This is equivalent to $3 \le y \le 5$.

Half-Open (Half-Closed) Interval: $x \in (-9, -2]$. This is equivalent to $-9 < x \le -2$.

Infinite Interval: $y \in (2, \infty)$. This is equivalent to $y > 2$.

Another Infinite Interval: $x \in (-\infty, 3]$. This is equivalent to $x \le 3$.

Even, Odd & Symmetry

A function is called **even** if $f(-x) = f(x)$. Even functions are symmetric about the *y*-axis.

A function is called **odd** if $f(-x) = -f(x)$. Odd functions are symmetric about the origin.

Many functions are neither!

To test for even/odd, find $f(-x)$, and see if you can manipulate the results to become $f(x)$ or $-f(x)$.

Piecewise-defined Functions

Sometimes, we want to take bits and pieces of several functions, and paste them together into a Frankenstein's Monster function…like this:

$$f(x) = \begin{cases} x^2 & x < -1 \\ 2x - 1 & x \ge 2 \end{cases}$$

This is a **piecewise-defined function**. It consists of two or more functions, each on its own restricted (non-overlapping) domain.

The most famous (perhaps even useful) piecewise-defined function is the **absolute value function**: $|x| = \begin{cases} -x & x < 0 \\ x & x \ge 0 \end{cases}$.

Composition of Functions

One fabulously important idea is that of plugging one function into another. This is called **composing functions**, and has two notations: $f(g(x))$ and $(f \circ g)(x)$. In both of these cases, $g(x)$ is being plugged into $f(x)$.

Unlike PreCalculus, we will not spend much (any) time worrying about the domain of a composition of functions…

1.3: Exponential Functions

A **power function** is of the form $p(x) = x^n$, where *n* is a real number ($n \in \mathbb{R}$).

An **exponential function** is of the form $t(x) = n^x$, where *n* is a positive real number ($n \in \mathbb{R}^+$).

Notice the difference?

You should most definitely know your rules of exponents…which I will not repeat here. Read your textbook!

One of the biggest applications of exponential functions is growth and decay problems. It turns out that exponential models work well for population growth (up to a certain point; for a certain range of years) and for radioactive decay. You should have seen this before; we'll talk more about it when we can put The Calculus to use.

There is a certain number that arose from the study of exponential functions…**Euler's Number**; the **Natural Number**, *e*. One definition of *e* is $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

1.4: Parametric Equations

In almost all of your time in algebra, you have dealt with two variables—one independent (*x*) and one dependent (*y*). However, that isn't the only way to do it! It is possible (and even better, in some cases) to define both *x* and *y* as functions of some third variable. This third variable is called a *parameter*, and is almost always *t* (for time). For example:

$$C(t) = \begin{cases} x = 2t \\ y = 4t + 5 \end{cases}$$

This is a **parametric equation**. It happens to be the equation of a line with slope 2 and *y*-intercept $(0, 5)$…well, I should explain. For parametric functions, you don't graph the parameter. You'd need 3D graph paper, which (technically) doesn't exist. Each point on the graph has a value of *t* associated with it, but you can't see what that value is (on the graph). You only graph *x* and *y*.

The benefit of parametric equations is that they can describe all sorts of weird and funky shapes that aren't functions…but the individual equations that give *x* and *y* are functions, which enables us to do calculus tricks.

However…parametric equations are not on the AB syllabus…so I'll not dwell on them.

1.5: Functions and Logarithms

One-to-One Functions

A function is **one-to-one** if every *x*-value is mapped to a single *y*-value, and vice versa. Graphically, this means that the function also passes the **horizontal line test**—if a horizontal line intersects the function at more than one point, then the function is not one-to-one.

Inverses

If every x is matched with exactly one y, and every y is matched with exactly one x, then you can go back and forth between x- and y-values easily. You can't do that if two different x-values are matched with the same y—starting at y, to which x do you return?

When a function is one-to-one, it has an **inverse function**—a function that maps y-values back to x-values. The way to think of it is that the inverse function "undoes" the function, giving you back what you put in the function in the first place.

The notation for the inverse of $f(x)$ is $f^{-1}(x)$. You must be careful—the negative one exponent only means "one over" or "reciprocal" for *numbers* (or equations that could be reduced to numbers). When a negative one exponent is put on a function, then it means inverse.

Since an inverse "undoes" a function, when a function and its inverse are composed, they wipe each other out: $f\left(f^{-1}(x)\right) = x$.

To find an inverse analytically, swap x and y in the equation, and solve for y. To find an inverse graphically, reflect the function across the line $y = x$.

Logarithmic Functions

If you bother to graph an exponential function, you'll notice that it is one-to-one—thus, it has an inverse. The inverse of an exponential function is a **logarithmic function**. An exponential function takes a power and returns the base raised to that power. A logarithmic function takes the base already raised to a power, and returns the power. For example, the logarithm of 10^3 is 3. The notation is $l(x) = \log_n(x)$.

There are two important logarithmic functions. The lesser of the two is the **common logarithm** (base 10): $c(x) = \log_{10}(x)$. The most important logarithm is the **natural logarithm** (base e): $n(x) = \log_e(x) = \ln(x)$. In fact, this logarithm is so important that a lot of people (high school textbooks being the big exception) just write $y = \log(x)$ for the natural logarithm.

There are a number of properties of logarithms that come from the properties of exponents. Again, I'll not give them here…read your textbook!

One formula that I will give is the change of base formula: $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$. You don't *have* to use natural logarithms, but it's just so…well, *natural*…

1.6: Trigonometric Functions

Angles

There are two important ways to measure angles. First, your favorite: degrees. I'm sure that you are intimately familiar with degree measures, so I'll say no more.

Second, the more important way to measure an angle: radians. Remember that one radian is the angle that subtends (cuts) an arc on a circle with an arc length equal to the radius. Most of the things that we will do will require that the angles be measured in radians; thus, unless told otherwise, assume that angles are measured in radians.

Trigonometric Functions

If you pick any point (x, y) in the coordinate plane, and connect it back to the origin, then that line and the positive *x*-axis form an angle—let's call it θ for old time's sake. Also, let *r* be the distance from (x, y) to the origin. In that case, we can then define…

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\csc\theta = \frac{r}{y}$, $\sec\theta = \frac{r}{x}$, $\cot\theta = \frac{x}{y}$

You can graph these functions…when you do, θ is plotted horizontally (*x*-axis) and the function value is plotted vertically. You should know what these graphs look like…they are in the textbook if you need a reminder.

Transformations

This actually applies to *any* function! Here is the (correct) template:

$t(x) = a \cdot f\left(b(x-c)\right) + d$

The graph of $t(x)$ can be found by taking the graph of $f(x)$ and doing the following:

- Multiply the *y*-values by *a*
- Divide the *x*-values by *b*
- Shift the graph *c* units horizontally (add *c* to every *x*-value)
- Shift the graph *d* units vertically (add *d* to every *y*-value)

For sine (and cosine) graphs, $|a|$ is the **amplitude**; $\frac{2\pi}{b}$ is the **period**; and *c* is the **phase shift**.

Inverse Trigonometric Functions

Looking at the graphs of the six basic trigonometric functions, you should see that none of them are one-to-one—which means that none of them have inverse functions.

This is unacceptable! Thus, we restrict the domains of the functions so that each has an inverse. Specifically, we restrict sine, tangent and cosecant to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We restrict cosine, cotangent and secant to the interval $[0, \pi]$. Read your textbook to see which functions use the closed interval and which must use the open interval.

The notation for the inverse trigonometric functions can be written in two ways: $\arcsin(x) = \sin^{-1}(x)$, for example. There are a few people out there that will still refer to yet another function, $\text{Arcsin}(x)$, which is not technically an inverse function…we won't be messing with that one; when we talk about the arc-functions, we're referring to the real inverses on the restricted domains.

Your calculator can do three of these inverse functions; to handle the others, you'll need to do a bit of algebra to get things in terms of a function that you can use.